CPT groups of higher spin fields

V. V. Varlamov*

Abstract

CPT groups of higher spin fields are defined in the framework of automorphism groups of Clifford algebras associated with the complex representations of the proper orthochronous Lorentz group. Higher spin fields are understood as the fields on the Poincaré group which describe orientable (extended) objects. A general method for construction of CPT groups of the fields of any spin is given. CPT groups of the fields of spin-1/2, spin-1 and spin-3/2 are considered in detail. CPT groups of the fields of tensor type are discussed. It is shown that tensor fields correspond to particles of the same spin with different masses.

Keywords: CPT groups, fields on the Poincaré group, Clifford algebras, automorphism groups, higher spin fields
PACS numbers: **02.10.Tq, 11.30.Er, 11.30.Cp**

1 Introduction

In 2003, CPT group was introduced [55] in the context of an extension of automorphism groups of Clifford algebras. The relationship between CPT groups and extraspecial groups and universal coverings of orthogonal groups was established in [55, 57]. In 2004, Socolovsky considered the CPT group of the spinor field with respect to phase quantities [50] (see also [10, 11, 12, 13, 37]). CPT groups of spinor fields in the de Sitter spaces of different signatures were studied in the works [58, 60]. The following logical step in this direction is a definition of the CPT groups for the higher spin fields. The formalism developed in the previous works [55, 57] allows us to define CPT groups for the fields of any spin on the spinspaces associated with representations of the spinor group $\mathbf{Spin}_+(1,3)$ (a universal covering of the proper orthochronous Lorentz group).

Our consideration based on the concept of generalized wavefunctions introduced by Ginzburg and Tamm in 1947 [22], where the wavefunction depends both coordinates x_μ and additional internal variables θ_μ which describe spin of the particle, $\mu = 0,1,2,3$. In 1955, Finkelstein showed [18] that elementary particles models with internal degrees of freedom can be described on manifolds larger then Minkowski spacetime (homogeneous spaces of the Poincaré group). The quantum field theories on the Poincaré group were discussed in the papers [33, 28, 5, 3, 30, 51, 34, 17, 23, 26]. A consideration of the field models on the homogeneous spaces leads naturally to a generalization of the concept of wave function (fields on the Poincaré group). The general form of these fields is related closely with the structure of the Lorentz and Poincaré group representations [21, 36, 4, 23] and admits the following factorization $f(x,\mathbf{z}) = \phi^n(\mathbf{z})\psi_n(x)$, where $x \in T_4$ and $\phi^n(\mathbf{z})$ form a basis in the representation space of the Lorentz group. At this point, four parameters x^μ correspond to position of the point-like object, whereas remaining six parameters $\mathbf{z} \in \mathbf{Spin}_+(1,3)$ define orientation in quantum description of orientable (extended) object [24, 25] (see also [27]). It is obvious

*Siberian State Industrial University, Kirova 42, Novokuznetsk 654007, Russia, e-mail: vadim.varlamov@mail.ru

that the point-like object has no orientation, therefore, orientation is an intrinsic property of the extended object. On the other hand, measurements in quantum field theory lead to extended objects. As is known, loop divergences emerging in the Green functions in quantum field theory originate from correspondence of the Green functions to *unmeasurable* (and hence unphysical) point-like quantities. This is because no physical quantity can be measured in a point, but in a region, the size of which (or 'diameter' of the extended object) is constrained by the resolution of measuring equipment [2]. Taking it into account, we come to consideration of physical quantity as an extended object, the generalized wavefunction of which is described by the field

$$\boldsymbol{\psi}(\boldsymbol{\alpha}) = \langle x, g\,|\boldsymbol{\psi}\rangle$$

on the homogeneous space of some orthogonal group $\mathrm{SO}(p,q)$, where $x \in T_n$ (position) and $g \in \mathbf{Spin}_+(p,q)$ (orientation), $n = p+q$. So, in [45, 46] Segal and Zhou proved convergence of quantum field theory, in particular, quantum electrodynamics, on the homogeneous space $R^1 \times S^3$ of the conformal group $\mathrm{SO}(2,4)$, where S^3 is the three-dimensional real sphere.

In the present work we describe discrete symmetries of the generalized wavefunctions $\boldsymbol{\psi}(\boldsymbol{\alpha}) = \langle x, \mathfrak{g}\,|\boldsymbol{\psi}\rangle$ (fields on the Poincaré group) in terms of involutive automorphisms of the subgroup $\mathbf{Spin}_+(p,q)$, As is known, the universal covering of the proper Poincaré group is isomorphic to a semidirect product $\mathrm{SL}(2;\mathbb{C}) \odot T_4$ or $\mathbf{Spin}_+(1,3) \odot T_4$. Since the group T_4 is Abelian, then all its representations are one-dimensional. Thus, all the finite-dimensional representations of the proper Poincaré group in essence are equivalent to the representations $\mathfrak{C}$ of the group $\mathbf{Spin}_+(1,3)$.

An algebraic method for description of discrete symmetries was proposed by author in the works [52, 53, 55, 57], where the discrete symmetries are represented by fundamental automorphisms of the Clifford algebras. So, the space inversion P, time reversal T and their combination PT correspond to an automorphism $\star$ (involution), an antiautomorphism $\widetilde{\ }$ (reversion) and an antiautomorphism $\widetilde{\star}$ (conjugation), respectively. The fundamental automorphisms of the Clifford algebras are compared to elements of the finite group formed by the discrete transformations. In turn, a set of the fundamental automorphisms, added by an identical automorphism, forms a finite group $\mathrm{Aut}(C\!\ell)$, for which in virtue of the Wedderburn-Artin Theorem there exists a matrix (spinor) representation. Further, other important discrete symmetry is the charge conjugation C. In contrast with the transformations P, T, PT, the operation C is not space-time discrete symmetry. This transformation is firstly appeared on the representation spaces of the Lorentz group and its nature is strongly different from other discrete symmetries. For that reason the charge conjugation C is represented by a pseudoautomorphism $\mathcal{A} \rightarrow \overline{\mathcal{A}}$ which is not fundamental automorphism of the Clifford algebra. All spinor representations of the pseudoautomorphism $\mathcal{A} \rightarrow \overline{\mathcal{A}}$ were given in [55]. An introduction of the transformation $\mathcal{A} \rightarrow \overline{\mathcal{A}}$ allows us to extend the automorphism group $\mathrm{Aut}(C\!\ell)$ of the Clifford algebra. It was shown [55] that automorphisms $\mathcal{A} \rightarrow \mathcal{A}^\star$, $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$, $\mathcal{A} \rightarrow \widetilde{\mathcal{A}^\star}$, $\mathcal{A} \rightarrow \overline{\mathcal{A}}$, $\mathcal{A} \rightarrow \overline{\mathcal{A}^\star}$, $\mathcal{A} \rightarrow \overline{\widetilde{\mathcal{A}}}$ and $\mathcal{A} \rightarrow \overline{\widetilde{\mathcal{A}^\star}}$ form a finite group of order 8 (an extended automorphism group $\mathrm{Ext}(C\!\ell) = \{\mathrm{Id}, \star, \widetilde{\ }, \widetilde{\star}, \overline{\ }, \overline{\star}, \overline{\widetilde{\ }}, \overline{\widetilde{\star}}\}$). The group $\mathrm{Ext}(C\!\ell)$ is a generating group of the full CPT group $\{\pm 1, \pm P, \pm T, \pm PT, \pm C, \pm CP, \pm CT, \pm CPT\}$. There are also other realizations of the discrete symmetries via the automorphisms of the Lorentz and Poincaré groups, see [21, 35, 29, 49, 9].

The present paper is organized as follows. In the section 2 we briefly discuss the basis notions concerning Clifford algebras and CPT groups, and also we consider their descriptions within universal coverings of orthogonal groups and spinor representations. In the section 3 we introduce the main objects of our study, CPT groups of higher spin fields. These groups are defined on the system of complex representations of the group $\mathbf{Spin}_+(1,3)$. In the sections 4–6 we consider in detail CPT groups for the fields $(1/2,0) \oplus (0,1/2)$, $(1,0) \oplus (0,1)$ and $(3/2,0) \oplus (0,3/2)$. In the section 7 we define CPT groups for the fields of tensor type.

2 Algebraic and group theoretical preliminaries

In this section we will consider some basic facts concerning automorphisms of the Clifford algebras and universal coverings of orthogonal groups.

Let $\mathbb{F}$ be a field of characteristic 0 ($\mathbb{F}=\mathbb{R}$, $\mathbb{F}=\mathbb{C}$), where $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, respectively. A Clifford algebra $C\ell$ over a field $\mathbb{F}$ is an algebra with 2^n basis elements: $\mathbf{e}_0$ (unit of the algebra) $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n$ and products of the one–index elements $\mathbf{e}_{i_1i_2\ldots i_k}=\mathbf{e}_{i_1}\mathbf{e}_{i_2}\ldots\mathbf{e}_{i_k}$. Over the field $\mathbb{F}=\mathbb{R}$ the Clifford algebra is denoted as $C\ell_{p,q}$, where the indices p and q correspond to the indices of the quadratic form

$$Q=x_1^2+\ldots+x_p^2-\ldots-x_{p+q}^2$$

of a vector space V associated with $C\ell_{p,q}$.

An arbitrary element $\mathcal{A}$ of the algebra $C\ell_{p,q}$ is represented by a following formal polynomial:

$$\mathcal{A}=a^0\mathbf{e}_0+\sum_{i=1}^n a^i\mathbf{e}_i+\sum_{i=1}^n\sum_{j=1}^n a^{ij}\mathbf{e}_{ij}+\ldots+\sum_{i_1=1}^n\cdots\sum_{i_k=1}^n a^{i_1\ldots i_k}\mathbf{e}_{i_1\ldots i_k}+$$
$$+\ldots+a^{12\ldots n}\mathbf{e}_{12\ldots n}=\sum_{k=0}^n a^{i_1i_2\ldots i_k}\mathbf{e}_{i_1i_2\ldots i_k}.$$

In Clifford algebra $C\ell$ there exist four fundamental automorphisms.

1) **Identity**: An automorphism $\mathcal{A}\rightarrow\mathcal{A}$ and $\mathbf{e}_i\rightarrow\mathbf{e}_i$.
This automorphism is an identical automorphism of the algebra $C\ell$. $\mathcal{A}$ is an arbitrary element of $C\ell$.

2) **Involution**: An automorphism $\mathcal{A}\rightarrow\mathcal{A}^\star$ and $\mathbf{e}_i\rightarrow-\mathbf{e}_i$.
In more details, for an arbitrary element $\mathcal{A}\in C\ell$ there exists a decomposition $\mathcal{A}=\mathcal{A}'+\mathcal{A}''$, where $\mathcal{A}'$ is an element consisting of homogeneous odd elements, and $\mathcal{A}''$ is an element consisting of homogeneous even elements, respectively. Then the automorphism $\mathcal{A}\rightarrow\mathcal{A}^\star$ is such that the element $\mathcal{A}''$ is not changed, and the element $\mathcal{A}'$ changes sign: $\mathcal{A}^\star=-\mathcal{A}'+\mathcal{A}''$. If $\mathcal{A}$ is a homogeneous element, then

$$\mathcal{A}^\star=(-1)^k\mathcal{A}, \tag{1}$$

where k is a degree of the element. It is easy to see that the automorphism $\mathcal{A}\rightarrow\mathcal{A}^\star$ may be expressed via the volume element $\omega=\mathbf{e}_{12\ldots p+q}$:

$$\mathcal{A}^\star=\omega\mathcal{A}\omega^{-1}, \tag{2}$$

where $\omega^{-1}=(-1)^{\frac{(p+q)(p+q-1)}{2}}\omega$. When k is odd, the basis elements $\mathbf{e}_{i_1i_2\ldots i_k}$ the sign changes, and when k is even, the sign is not changed.

3) **Reversion**: An antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}}$ and $\mathbf{e}_i\rightarrow\mathbf{e}_i$.
The antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}}$ is a reversion of the element $\mathcal{A}$, that is the substitution of each basis element $\mathbf{e}_{i_1i_2\ldots i_k}\in\mathcal{A}$ by the element $\mathbf{e}_{i_ki_{k-1}\ldots i_1}$:

$$\mathbf{e}_{i_ki_{k-1}\ldots i_1}=(-1)^{\frac{k(k-1)}{2}}\mathbf{e}_{i_1i_2\ldots i_k}.$$

Therefore, for any $\mathcal{A}\in C\ell_{p,q}$ we have

$$\widetilde{\mathcal{A}}=(-1)^{\frac{k(k-1)}{2}}\mathcal{A}. \tag{3}$$

4) **Conjugation**: An antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}^\star}$ and $\mathbf{e}_i \to -\mathbf{e}_i$.
This antiautomorphism is a composition of the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ with the automorphism $\mathcal{A} \to \mathcal{A}^\star$. In the case of a homogeneous element from the formulae (1) and (3), it follows

$$\widetilde{\mathcal{A}^\star} = (-1)^{\frac{k(k+1)}{2}}\mathcal{A}. \tag{4}$$

As is known, the complex algebra $\mathbb{C}_n$ is associated with a complex vector space $\mathbb{C}^n$. Let $n = p+q$, then an extraction operation of the real subspace $\mathbb{R}^{p,q}$ in $\mathbb{C}^n$ forms the foundation of definition of the discrete transformation known in physics as *a charge conjugation* C. Indeed, let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be an orthobasis in the space $\mathbb{C}^n$, $\mathbf{e}_i^2 = 1$. Let us remain the first p vectors of this basis unchanged, and other q vectors multiply by the factor i. Then the basis

$$\{\mathbf{e}_1, \ldots, \mathbf{e}_p, i\mathbf{e}_{p+1}, \ldots, i\mathbf{e}_{p+q}\} \tag{5}$$

allows one to extract the subspace $\mathbb{R}^{p,q}$ in $\mathbb{C}^n$. Namely, for the vectors $\mathbb{R}^{p,q}$ we take the vectors of $\mathbb{C}^n$ which decompose on the basis (5) with real coefficients. In such a way we obtain a real vector space $\mathbb{R}^{p,q}$ endowed (in general case) with a non–degenerate quadratic form

$$Q(x) = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \ldots - x_{p+q}^2,$$

where $x_1, \ldots, x_{p+q}$ are coordinates of the vector $\mathbf{x}$ in the basis (5). It is easy to see that the extraction of $\mathbb{R}^{p,q}$ in $\mathbb{C}^n$ induces an extraction of *a real subalgebra* $C\!\ell_{p,q}$ in $\mathbb{C}_n$. Therefore, any element $\mathcal{A} \in \mathbb{C}_n$ can be unambiguously represented in the form

$$\mathcal{A} = \mathcal{A}_1 + i\mathcal{A}_2,$$

where $\mathcal{A}_1$, $\mathcal{A}_2 \in C\!\ell_{p,q}$. The one-to-one mapping

$$\mathcal{A} \longrightarrow \overline{\mathcal{A}} = \mathcal{A}_1 - i\mathcal{A}_2 \tag{6}$$

transforms the algebra $\mathbb{C}_n$ into itself with preservation of addition and multiplication operations for the elements $\mathcal{A}$; the operation of multiplication of the element $\mathcal{A}$ by the number transforms to an operation of multiplication by the complex conjugate number. Any mapping of $\mathbb{C}_n$ satisfying these conditions is called *a pseudoautomorphism*. Thus, the extraction of the subspace $\mathbb{R}^{p,q}$ in the space $\mathbb{C}^n$ induces in the algebra $\mathbb{C}_n$ a pseudoautomorphism $\mathcal{A} \to \overline{\mathcal{A}}$ [39, 40].

An introduction of the pseudoautomorphism $\mathcal{A} \to \overline{\mathcal{A}}$ allows us to extend the automorphism set of the complex Clifford algebra $\mathbb{C}_n$. Namely, we add to the four fundamental automorphisms $\mathcal{A} \to \mathcal{A}$, $\mathcal{A} \to \mathcal{A}^\star$, $\mathcal{A} \to \widetilde{\mathcal{A}}$, $\mathcal{A} \to \widetilde{\mathcal{A}^\star}$ the pseudoautomorphism $\mathcal{A} \to \overline{\mathcal{A}}$ and following three combinations:
1) A pseudoautomorphism $\mathcal{A} \to \overline{\mathcal{A}^\star}$. This transformation is a composition of the pseudoautomorphism $\mathcal{A} \to \overline{\mathcal{A}}$ with the automorphism $\mathcal{A} \to \mathcal{A}^\star$.
2) A pseudoantiautomorphism $\mathcal{A} \to \overline{\widetilde{\mathcal{A}}}$. This transformation is a composition of $\mathcal{A} \to \overline{\mathcal{A}}$ with the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$.
3) A pseudoantiautomorphism $\mathcal{A} \to \overline{\widetilde{\mathcal{A}^\star}}$ (a composition of $\mathcal{A} \to \overline{\mathcal{A}}$ with the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}^\star}$).

Thus, we obtain an automorphism set of $\mathbb{C}_n$ consisting of the eight transformations. Let us show that the set $\{\mathrm{Id}, \star, \widetilde{}, \widetilde{\star}, \overline{}, \overline{\star}, \overline{\widetilde{}}, \overline{\widetilde{\star}}\}$ forms a finite group of order 8 and let us give a physical interpretation of this group.

Proposition 1 ([55]). *Let $\mathbb{C}_n$ be a Clifford algebra over the field $\mathbb{F} = \mathbb{C}$ and let $\mathrm{Ext}(\mathbb{C}_n) = \{\mathrm{Id}, \star, \widetilde{}, \widetilde{\star}, \overline{}, \overline{\star}, \overline{\widetilde{}}, \overline{\widetilde{\star}}\}$ be an extended automorphism group of the algebra $\mathbb{C}_n$. Then there is*

an isomorphism between $\mathrm{Ext}(\mathbb{C}_n)$ *and* $CPT/\mathbb{Z}_2$ *group of the discrete transformations,* $\mathrm{Ext}(\mathbb{C}_n) \simeq \{1, P, T, PT, C, CP, CT, CPT\} \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$. *In this case, space inversion* P*, time reversal* T*, full reflection* PT*, charge conjugation* C*, transformations* CP*,* CT *and the full* CPT*–transformation correspond to the automorphism* $\mathcal{A} \rightarrow \mathcal{A}^\star$*, antiautomorphisms* $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$*,* $\mathcal{A} \rightarrow \widetilde{\mathcal{A}^\star}$*, pseudoautomorphisms* $\mathcal{A} \rightarrow \overline{\mathcal{A}}$*,* $\mathcal{A} \rightarrow \overline{\mathcal{A}^\star}$*, pseudoantiautomorphisms* $\mathcal{A} \rightarrow \overline{\widetilde{\mathcal{A}}}$ *and* $\mathcal{A} \rightarrow \overline{\widetilde{\mathcal{A}^\star}}$*, respectively.*

Proof. The group $\{1, P, T, PT, C, CP, CT, CPT\}$ at the conditions $P^2 = T^2 = (PT)^2 = C^2 = (CP)^2 = (CT)^2 = (CPT)^2 = 1$ and commutativity of all the elements forms an Abelian group of order 8, which is isomorphic to a cyclic group $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$. The multiplication table of this group shown in Tab. 1.

	1	P	T	PT	C	CP	CT	CPT
1	1	P	T	PT	C	CP	CT	CPT
P	P	1	PT	T	CP	C	CPT	CT
T	T	PT	1	P	CT	CPT	C	CP
PT	PT	T	P	1	CPT	CT	CP	C
C	C	CP	CT	CPT	1	P	T	PT
CP	CP	C	CPT	CT	P	1	PT	T
CT	CT	CPT	C	CP	T	PT	1	P
CPT	CPT	CT	CP	C	PT	T	P	1

Tab. 1: The multiplication table of the $CPT/\mathbb{Z}_2$ group.

In turn, for the extended automorphism group $\{\mathrm{Id}, \star, \widetilde{\ }, \widetilde{\star}, \overline{\ }, \overline{\star}, \overline{\widetilde{\ }}, \overline{\widetilde{\star}}\}$ in virtue of commutativity $\widetilde{(\mathcal{A}^\star)} = \left(\widetilde{\mathcal{A}}\right)^\star$, $\overline{(\mathcal{A}^\star)} = (\overline{\mathcal{A}})^\star$, $\overline{\left(\widetilde{\mathcal{A}}\right)} = \widetilde{(\overline{\mathcal{A}})}$, $\overline{\left(\widetilde{\mathcal{A}^\star}\right)} = \widetilde{(\overline{\mathcal{A}})}^\star$ and an involution property $\star\star = \widetilde{\ }\,\widetilde{\ } = \overline{\ }\,\overline{\ } = \mathrm{Id}$ we have the multiplication table shown in Tab. 2. The identity of multipli-

	Id	$\star$	$\widetilde{\ }$	$\widetilde{\star}$	$\overline{\ }$	$\overline{\star}$	$\overline{\widetilde{\ }}$	$\overline{\widetilde{\star}}$
Id	Id	$\star$	$\widetilde{\ }$	$\widetilde{\star}$	$\overline{\ }$	$\overline{\star}$	$\overline{\widetilde{\ }}$	$\overline{\widetilde{\star}}$
$\star$	$\star$	Id	$\widetilde{\star}$	$\widetilde{\ }$	$\overline{\star}$	$\overline{\ }$	$\overline{\widetilde{\star}}$	$\overline{\widetilde{\ }}$
$\widetilde{\ }$	$\widetilde{\ }$	$\overline{\star}$	Id	$\star$	$\overline{\widetilde{\ }}$	$\overline{\widetilde{\star}}$	$\overline{\ }$	$\overline{\star}$
$\widetilde{\star}$	$\widetilde{\star}$	$\widetilde{\ }$	$\star$	Id	$\overline{\widetilde{\star}}$	$\overline{\widetilde{\ }}$	$\overline{\star}$	$\overline{\ }$
$\overline{\ }$	$\overline{\ }$	$\overline{\star}$	$\overline{\widetilde{\ }}$	$\overline{\widetilde{\star}}$	Id	$\star$	$\widetilde{\ }$	$\widetilde{\star}$
$\overline{\star}$	$\overline{\star}$	$\overline{\ }$	$\overline{\widetilde{\star}}$	$\overline{\widetilde{\ }}$	$\star$	Id	$\widetilde{\star}$	$\widetilde{\ }$
$\overline{\widetilde{\ }}$	$\overline{\widetilde{\ }}$	$\overline{\widetilde{\star}}$	$\overline{\ }$	$\overline{\star}$	$\widetilde{\ }$	$\widetilde{\star}$	Id	$\star$
$\overline{\widetilde{\star}}$	$\overline{\widetilde{\star}}$	$\overline{\widetilde{\ }}$	$\overline{\star}$	$\overline{\ }$	$\widetilde{\star}$	$\widetilde{\ }$	$\star$	Id

Tab. 2: The multiplication table of the extended automorphism group.

cation tables proves the group isomorphism

$$\{1,\, P,\, T,\, PT,\, C,\, CP,\, CT,\, CPT\} \simeq \{\mathrm{Id},\, \star,\, \widetilde{\ },\, \widetilde{\star},\, \overline{\ },\, \overline{\star},\, \overline{\widetilde{\ }},\, \overline{\widetilde{\star}}\} \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2.$$

□

Further, in the case of $P^2 = T^2 = \ldots = (CPT)^2 = \pm 1$ and anticommutativity of the elements we have an isomorphism between the $CPT/\mathbb{Z}_2$ group and a group $\mathsf{Ext}(\mathbb{C}_n)$. The elements of $\mathsf{Ext}(\mathbb{C}_n)$ are spinor representations of the automorphisms of the algebra $\mathbb{C}_n$. As mentioned previously, the Wedderburn-Artin Theorem allows us to define any spinor representaions for the automorphisms of $\mathbb{C}_n$. We list these transformations and their spinor representations (for more details see [55]):

$$\mathcal{A} \longrightarrow \mathcal{A}^\star, \qquad \mathsf{A}^\star = \mathsf{W}\mathsf{A}\mathsf{W}^{-1}, \tag{7}$$

$$\mathcal{A} \longrightarrow \widetilde{\mathcal{A}}, \qquad \widetilde{\mathsf{A}} = \mathsf{E}\mathsf{A}^{\mathsf{T}}\mathsf{E}^{-1}, \tag{8}$$

$$\mathcal{A} \longrightarrow \widetilde{\mathcal{A}^\star}, \qquad \widetilde{\mathsf{A}^\star} = \mathsf{C}\mathsf{A}^{\mathsf{T}}\mathsf{C}^{-1}, \quad \mathsf{C} = \mathsf{E}\mathsf{W}, \tag{9}$$

$$\mathcal{A} \longrightarrow \overline{\mathcal{A}}, \qquad \overline{\mathsf{A}} = \Pi\mathsf{A}^{*}\Pi^{-1}, \tag{10}$$

$$\mathcal{A} \longrightarrow \overline{\mathcal{A}^\star}, \qquad \overline{\mathsf{A}^\star} = \mathsf{K}\mathsf{A}^{*}\mathsf{K}^{-1}, \quad \mathsf{K} = \Pi\mathsf{W}, \tag{11}$$

$$\mathcal{A} \longrightarrow \overline{\widetilde{\mathcal{A}}}, \qquad \overline{\widetilde{\mathsf{A}}} = \mathsf{S}\left(\mathsf{A}^{\mathsf{T}}\right)^{*}\mathsf{S}^{-1}, \quad \mathsf{S} = \Pi\mathsf{E}, \tag{12}$$

$$\mathcal{A} \longrightarrow \overline{\widetilde{\mathcal{A}^\star}}, \qquad \overline{\widetilde{\mathsf{A}^\star}} = \mathsf{F}\left(\mathsf{A}^{*}\right)^{\mathsf{T}}\mathsf{F}^{-1}, \quad \mathsf{F} = \Pi\mathsf{C}, \tag{13}$$

where the symbol T means a transposition, and $*$ is a complex conjugation. The detailed classification of the extended automorphism groups $\mathsf{Ext}(\mathbb{C}_n)$ was given in [55]. First of all, since for the subalgebras $C\ell_{p,q}$ over the ring $\mathbb{K} \simeq \mathbb{R}$ the group $\mathsf{Ext}(\mathbb{C}_n)$ is reduced to $\mathsf{Aut}_\pm(\mathbb{C}_n)$ (reflection group [53]), then all the essentially different groups $\mathsf{Ext}(\mathbb{C}_n)$ correspond to subalgebras $C\ell_{p,q}$ with the quaternionic ring $\mathbb{K} \simeq \mathbb{H}$, $p - q \equiv 4, 6 \pmod 8$. The classification of the groups $\mathsf{Ext}(\mathbb{C}_n)$ is given with respect to the subgroups $\mathsf{Aut}_\pm(C\ell_{p,q})$. Taking into account the structure of $\mathsf{Aut}_\pm(C\ell_{p,q})$, we have at $p - q \equiv 4, 6 \pmod 8$ for the groups $\mathsf{Ext}(\mathbb{C}_n) = \{\mathsf{I}, \mathsf{W}, \mathsf{E}, \mathsf{C}, \Pi, \mathsf{K}, \mathsf{S}, \mathsf{F}\}$ the following realizations [55]:

$$\begin{aligned}
\mathsf{Ext}^1(\mathbb{C}_n) &= \left\{\mathsf{I}, \mathcal{E}_{12\cdots p+q}, \mathcal{E}_{j_1 j_2\cdots j_k}, \mathcal{E}_{i_1 i_2\cdots i_{p+q-k}}, \mathcal{E}_{\alpha_1\alpha_2\cdots\alpha_a}, \mathcal{E}_{\beta_1\beta_2\cdots\beta_b}, \mathcal{E}_{c_1 c_2\cdots c_s}, \mathcal{E}_{d_1 d_2\cdots d_g}\right\}, \\
\mathsf{Ext}^2(\mathbb{C}_n) &= \left\{\mathsf{I}, \mathcal{E}_{12\cdots p+q}, \mathcal{E}_{j_1 j_2\cdots j_k}, \mathcal{E}_{i_1 i_2\cdots i_{p+q-k}}, \mathcal{E}_{\beta_1\beta_2\cdots\beta_b}, \mathcal{E}_{\alpha_1\alpha_2\cdots\alpha_a}, \mathcal{E}_{d_1 d_2\cdots d_g}, \mathcal{E}_{c_1 c_2\cdots c_s}\right\}, \\
\mathsf{Ext}^3(\mathbb{C}_n) &= \left\{\mathsf{I}, \mathcal{E}_{12\cdots p+q}, \mathcal{E}_{i_1 i_2\cdots i_{p+q-k}}, \mathcal{E}_{j_1 j_2\cdots j_k}, \mathcal{E}_{\alpha_1\alpha_2\cdots\alpha_a}, \mathcal{E}_{\beta_1\beta_2\cdots\beta_b}, \mathcal{E}_{d_1 d_2\cdots d_g}, \mathcal{E}_{c_1 c_2\cdots c_s}\right\}, \\
\mathsf{Ext}^4(\mathbb{C}_n) &= \left\{\mathsf{I}, \mathcal{E}_{12\cdots p+q}, \mathcal{E}_{i_1 i_2\cdots i_{p+q-k}}, \mathcal{E}_{j_1 j_2\cdots j_k}, \mathcal{E}_{\beta_1\beta_2\cdots\beta_b}, \mathcal{E}_{\alpha_1\alpha_2\cdots\alpha_a}, \mathcal{E}_{c_1 c_2\cdots c_s}, \mathcal{E}_{d_1 d_2\cdots d_g}\right\}.
\end{aligned}$$

The groups $\mathsf{Ext}^1(\mathbb{C}_n)$ and $\mathsf{Ext}^2(\mathbb{C}_n)$ have Abelian subgroups $\mathsf{Aut}_-(C\ell_{p,q})$ ($\mathbb{Z}_2 \otimes \mathbb{Z}_2$ or $\mathbb{Z}_4$). In turn, the groups $\mathsf{Ext}^3(\mathbb{C}_n)$ and $\mathsf{Ext}^4(\mathbb{C}_n)$ have non-Abelian subgroups $\mathsf{Aut}_+(C\ell_{p,q})$ ($Q_4/\mathbb{Z}_2$ or $D_4/\mathbb{Z}_2$). The full number of different realizations of $\mathsf{Ext}(\mathbb{C}_n)$ is 64.

As is known, the Lipschitz group $\mathbf{\Gamma}_{p,q}$, also called the Clifford group, introduced by Lipschitz in 1886 [31], may be defined as the subgroup of invertible elements s of the algebra $C\ell_{p,q}$:

$$\mathbf{\Gamma}_{p,q} = \left\{s \in C\ell^+_{p,q} \cup C\ell^-_{p,q} \mid \forall \mathbf{x} \in \mathbb{R}^{p,q},\ s\mathbf{x}s^{-1} \in \mathbb{R}^{p,q}\right\}.$$

The set $\mathbf{\Gamma}^+_{p,q} = \mathbf{\Gamma}_{p,q} \cap C\ell^+_{p,q}$ is called *special Lipschitz group* [14].

Let $N:\ C\ell_{p,q} \to C\ell_{p,q}$, $N(\mathbf{x}) = \mathbf{x}\widetilde{\mathbf{x}}$. If $\mathbf{x} \in \mathbb{R}^{p,q}$, then $N(\mathbf{x}) = \mathbf{x}(-\mathbf{x}) = -\mathbf{x}^2 = -Q(\mathbf{x})$. Further, the group $\mathbf{\Gamma}_{p,q}$ has a subgroup

$$\mathbf{Pin}(p,q) = \{s \in \mathbf{\Gamma}_{p,q} \mid N(s) = \pm 1\}. \tag{14}$$

Analogously, *a spinor group* $\mathbf{Spin}(p,q)$ is defined by the set

$$\mathbf{Spin}(p,q) = \left\{ s \in \mathbf{\Gamma}^{+}_{p,q} \mid N(s) = \pm 1 \right\}. \tag{15}$$

It is obvious that $\mathbf{Spin}(p,q) = \mathbf{Pin}(p,q) \cap C\ell^{+}_{p,q}$. The group $\mathbf{Spin}(p,q)$ contains a subgroup

$$\mathbf{Spin}_{+}(p,q) = \{ s \in \mathbf{Spin}(p,q) \mid N(s) = 1 \}. \tag{16}$$

The groups $\mathrm{O}(p,q)$, $\mathrm{SO}(p,q)$ and $\mathrm{SO}_{+}(p,q)$ are isomorphic, respectively, to the following quotient groups

$$\mathrm{O}(p,q) \simeq \mathbf{Pin}(p,q)/\mathbb{Z}_2, \quad \mathrm{SO}(p,q) \simeq \mathbf{Spin}(p,q)/\mathbb{Z}_2, \quad \mathrm{SO}_{+}(p,q) \simeq \mathbf{Spin}_{+}(p,q)/\mathbb{Z}_2,$$

where the kernel $\mathbb{Z}_2 = \{1, -1\}$. Thus, the groups $\mathbf{Pin}(p,q)$, $\mathbf{Spin}(p,q)$ and $\mathbf{Spin}_{+}(p,q)$ are the universal coverings of the groups $\mathrm{O}(p,q)$, $\mathrm{SO}(p,q)$ and $\mathrm{SO}_{+}(p,q)$, respectively.

Over the field $\mathbb{F} = \mathbb{R}$ there exist 64 universal coverings of the real orthogonal group $\mathrm{O}(p,q)$:

$$\rho^{a,b,c,d,e,f,g} : \ \mathbf{Pin}^{a,b,c,d,e,f,g} \longrightarrow \mathrm{O}(p,q),$$

where

$$\mathbf{Pin}^{a,b,c,d,e,f,g}(p,q) \simeq \frac{(\mathbf{Spin}_{+}(p,q) \odot C^{a,b,c,d,e,f,g})}{\mathbb{Z}_2}, \tag{17}$$

and

$$C^{a,b,c,d,e,f,g} = \{\pm 1, \pm P, \pm T, \pm PT, \pm C, \pm CP, \pm CT, \pm CPT\}$$

is *a full* CPT *group* [55, 57]. $C^{a,b,c,d,e,f,g}$ is a finite group of order 16. The group

$$\mathrm{Ext}(C\ell_{p,q}) = \frac{C^{a,b,c,d,e,f,g}}{\mathbb{Z}_2} \simeq CPT/\mathbb{Z}_2$$

is called *the generating group*. In essence, $C^{a,b,c,d,e,f,g}$ are five double coverings of the group $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$ (extraspecial Salingaros groups, see [43, 6]). All the possible double coverings $C^{a,b,c,d,e,f,g}$ are given in the Table 3. The group (17) with non-Abelian $C^{a,b,c,d,e,f,g}$ is called *Cliffordian group*

$a\ b\ c\ d\ e\ f\ g$	$C^{a,b,c,d,e,f,g}$	Type
$+\ +\ +\ +\ +\ +\ +$	$\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$	Abelian
three '+' and four '−'	$\mathbb{Z}_4 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$	
one '+' and six '−'	$Q_4 \otimes \mathbb{Z}_2$	Non–Abelian
five '+' and two '−'	$D_4 \otimes \mathbb{Z}_2$	
three '+' and four '−'	$\overset{*}{\mathbb{Z}}_4 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$	

Tab. 3: Extraspecial finite groups $C^{a,b,c,d,e,f,g}$ of order 16.

and respectively *non-Cliffordian group* when $C^{a,b,c,d,e,f,g}$ is Abelian. It is easy to see that in the case of the algebra $C\ell_{p,q}$ (or subalgebra $C\ell_{p,q} \subset \mathbb{C}_n$) with the real division ring $\mathbb{K} \simeq \mathbb{R}$, $p - q \equiv 0, 2 \pmod 8$, CPT-structures, defined by the groups (17), are reduced to the eight Shirokov-Dąbrowski PT-structures [47, 48, 16].

3 CPT groups on the representation spaces of $\mathbf{Spin}_+(1,3)$

Let us consider the field

$$\boldsymbol{\psi}(\boldsymbol{\alpha}) = \langle x, \mathfrak{g} \,|\boldsymbol{\psi}\rangle, \tag{18}$$

where $x \in T_4$, $\mathfrak{g} \in \mathbf{Spin}_+(1,3)$. The spinor group $\mathbf{Spin}_+(1,3) \simeq \mathrm{SU}(2)\otimes\mathrm{SU}(2)$ is a universal covering of the proper orthochronous Lorentz group $\mathrm{SO}_0(1,3)$. The parameters $x \in T_4$ and $\mathfrak{g} \in \mathbf{Spin}_+(1,3)$ describe position and orientation of the extended object defined by the field (18) (the field on the Poincaré group). The basic idea is to define discrete symmetries of the field (18) within the group

$$\mathbf{Pin}^{a,b,c,d,e,f,g}(1,3) \simeq \frac{\mathbf{Spin}_+(1,3)\odot C^{a,b,c,d,e,f,g}}{\mathbb{Z}_2}.$$

The automorphisms (discrete symmetries) of $\mathbf{Pin}^{a,b,c,d,e,f,g}(1,3)$ are outer automorphisms with respect to transformations of the group $\mathbf{Spin}_+(1,3)$. We define CPT groups $C^{a,b,c,d,e,f,g}$ of physical fields of any spin on the representation spaces of $\mathbf{Spin}_+(1,3)$.

Theorem 1. *Let*

$$\mathbf{Pin}^{a,b,c,d,e,f,g}(1,3) \simeq \frac{\mathbf{Spin}_+(1,3)\odot C^{a,b,c,d,e,f,g}}{\mathbb{Z}_2}$$

be the universal covering of the proper Lorentz group $\mathrm{SO}(1,3)$*, where* $C^{a,b,c,d,e,f,g} = \{\pm 1, \pm P, \pm T, \pm PT, \pm C, \pm CP, \pm CT, \pm CPT\}$ *is a* CPT *group of some physical field defined in the framework of finite-dimensional representation of the group* $\mathbf{Spin}_+(1,3)$*. At this point, there exits a correspondence* $P \sim \mathsf{W}$*,* $T \sim \mathsf{E}$*,* $PT \sim \mathsf{C}$*,* $C \sim \Pi$*,* $CP \sim \mathsf{K}$*,* $CT \sim \mathsf{S}$*,* $CPT \sim \mathsf{F}$*, where* $\{\mathsf{I}, \mathsf{W}, \mathsf{E}, \mathsf{C}, \Pi, \mathsf{K}, \mathsf{S}, \mathsf{F}\} \simeq \mathrm{Ext}(\mathbb{C}_n)$ *is an automorphism group of the algebra* $\mathbb{C}_n$*. Then* CPT *group of the field* $(l,0)\oplus(0,\dot{l})$ *is constructed in the framework of the finite-dimensional representation* $\mathfrak{C}^{l_0+l_1-1,0}\oplus\mathfrak{C}^{0,l_0-l_1+1}$ *of* $\mathbf{Spin}_+(1,3)$ *defined on the spinspace* $\mathbb{S}_{2^k}\otimes\mathbb{S}_{2^r}$ *with the algebra*

$$\underbrace{\mathbb{C}_2\otimes\mathbb{C}_2\otimes\cdots\otimes\mathbb{C}_2}_{k\ times}\bigoplus\underbrace{\overset{*}{\mathbb{C}}_2\otimes\overset{*}{\mathbb{C}}_2\otimes\cdots\otimes\overset{*}{\mathbb{C}}_2}_{r\ times},$$

where $(l_0,l_1) = \left(\frac{k}{2}, \frac{k}{2}+1\right)$*,* $(-l_0,l_1) = \left(-\frac{r}{2}, \frac{r}{2}+1\right)$*. In turn, a* CPT *group of the field* $(l',l'')\oplus(\dot{l}'',\dot{l}')$ *is constructed in the framework of representation* $\mathfrak{C}^{l_0+l_1-1,l_0-l_1+1}\oplus\mathfrak{C}^{l_0-l_1+1,l_0+l_1-1}$ *of* $\mathbf{Spin}_+(1,3)$ *defined on the spinspace* $\mathbb{S}_{2^{k+r}}\oplus\mathbb{S}_{2^{k+r}}$ *with the algebra*

$$\underbrace{\mathbb{C}_2\otimes\mathbb{C}_2\otimes\cdots\otimes\mathbb{C}_2\bigotimes\overset{*}{\mathbb{C}}_2\otimes\overset{*}{\mathbb{C}}_2\otimes\cdots\otimes\overset{*}{\mathbb{C}}_2}_{k+r\ times}\bigoplus\underbrace{\overset{*}{\mathbb{C}}_2\otimes\overset{*}{\mathbb{C}}_2\otimes\cdots\otimes\overset{*}{\mathbb{C}}_2\bigotimes\mathbb{C}_2\otimes\mathbb{C}_2\otimes\cdots\otimes\mathbb{C}_2}_{r+k\ times},$$

where $(l_0,l_1) = \left(\frac{k-r}{2}, \frac{k+r}{2}+1\right)$*.*

Proof. As is known, when $\mathcal{C}\!\ell_{p,q}$ is simple, then the map

$$\mathcal{C}\!\ell_{p,q} \overset{\gamma}{\longrightarrow} \mathrm{End}_{\mathbb{K}}(\mathbb{S}), \quad u \longrightarrow \gamma(u), \quad \gamma(u)\psi = u\psi \tag{19}$$

gives an irreducible and faithful representation of $\mathcal{C}\!\ell_{p,q}$ in the spinspace $\mathbb{S}_{2^m}(\mathbb{K}) \simeq I_{p,q} = \mathcal{C}\!\ell_{p,q}f$, where $\psi \in \mathbb{S}_{2^m}$, $m = \frac{p+q}{2}$.

On the other hand, when $\mathcal{C}\!\ell_{p,q}$ is semi-simple, then the map

$$\mathcal{C}\!\ell_{p,q} \overset{\gamma}{\longrightarrow} \mathrm{End}_{\mathbb{K}\oplus\hat{\mathbb{K}}}(\mathbb{S}\oplus\hat{\mathbb{S}}), \quad u \longrightarrow \gamma(u), \quad \gamma(u)\psi = u\psi \tag{20}$$

gives a faithful but reducible representation of $\mathcal{C}\!\ell_{p,q}$ in the double spinspace $\mathbb{S}\oplus\hat{\mathbb{S}}$, where $\hat{\mathbb{S}} = \{\hat{\psi} | \psi \in \mathbb{S}\}$. In this case, the ideal $\mathbb{S}\oplus\hat{\mathbb{S}}$ possesses a right $\mathbb{K}\oplus\hat{\mathbb{K}}$-linear structure, $\hat{\mathbb{K}} = \{\hat{\lambda} | \lambda \in \mathbb{K}\}$,

and $\mathbb{K}\oplus\hat{\mathbb{K}}$ is isomorphic to the double division ring $\mathbb{R}\oplus\mathbb{R}$ when $p-q\equiv 1 \pmod 8$ or to $\mathbb{H}\oplus\mathbb{H}$ when $p-q\equiv 5 \pmod 8$. The map γ in (19) and (20) defines the so called *left-regular* spinor representation of $C\!\ell(Q)$ in $\mathbb{S}$ and $\mathbb{S}\oplus\hat{\mathbb{S}}$, respectively. Furthermore, γ is *faithful* which means that γ is an algebra monomorphism. In (19), γ is *irreducible* which means that $\mathbb{S}$ possesses no proper (that is, $\neq 0$, $\mathbb{S}$) invariant subspaces under the left action of $\gamma(u)$, $u\in C\!\ell_{p,q}$. Representation γ in (20) is therefore *reducible* since $\{(\psi,0)|\psi\in\mathbb{S}\}$ and $\{(0,\hat{\psi})|\hat{\psi}\in\hat{\mathbb{S}}\}$ are two proper subspaces of $\mathbb{S}\oplus\hat{\mathbb{S}}$ invariant under $\gamma(u)$ (see [32, 15, 38]).

Since the spacetime algebra $C\!\ell_{1,3}$ is the simple algebra, then the map (19) gives an irreducible representation of $C\!\ell_{1,3}$ in the spinspace $\mathbb{S}_2(\mathbb{H})$. In turn, representations of the group $\mathbf{Spin}_+(1,3)\in C\!\ell^+_{1,3}\simeq C\!\ell_{3,0}$ are defined in the spinspace $\mathbb{S}_2(\mathbb{C})$.

Let us consider now spintensor representations of the group $\mathfrak{G}_+\simeq\mathrm{SL}(2;\mathbb{C})$ which, as is known, form the base of all the finite-dimensional representations of the Lorentz group, and also we consider their relationship with the complex Clifford algebras. From each complex Clifford algebra $\mathbb{C}_n=\mathbb{C}\otimes C\!\ell_{p,q}$ $(n=p+q)$ we obtain the spinspace $\mathbb{S}_{2^{n/2}}$ which is a complexification of the minimal left ideal of the algebra $C\!\ell_{p,q}$: $\mathbb{S}_{2^{n/2}}=\mathbb{C}\otimes I_{p,q}=\mathbb{C}\otimes C\!\ell_{p,q}f_{pq}$, where f_{pq} is the primitive idempotent of the algebra $C\!\ell_{p,q}$. Further, a spinspace related with the Pauli algebra $\mathbb{C}_2$ has the form $\mathbb{S}_2=\mathbb{C}\otimes I_{2,0}=\mathbb{C}\otimes C\!\ell_{2,0}f_{20}$ or $\mathbb{S}_2=\mathbb{C}\otimes I_{1,1}=\mathbb{C}\otimes C\!\ell_{1,1}f_{11}(\mathbb{C}\otimes I_{0,2}=\mathbb{C}\otimes C\!\ell_{0,2}f_{02})$. Therefore, the tensor product of the k algebras $\mathbb{C}_2$ induces a tensor product of the k spinspaces $\mathbb{S}_2$:

$$\mathbb{S}_2\otimes\mathbb{S}_2\otimes\cdots\otimes\mathbb{S}_2=\mathbb{S}_{2^k}.$$

Vectors of the spinspace $\mathbb{S}_{2^k}$ (or elements of the minimal left ideal of $\mathbb{C}_{2k}$) are spintensors of the following form:

$$\boldsymbol{s}^{\alpha_1\alpha_2\cdots\alpha_k}=\sum\boldsymbol{s}^{\alpha_1}\otimes\boldsymbol{s}^{\alpha_2}\otimes\cdots\otimes\boldsymbol{s}^{\alpha_k}, \tag{21}$$

where summation is produced on all the index collections $(\alpha_1\ldots\alpha_k)$, $\alpha_i=1,2$. For the each spinor $\boldsymbol{s}^{\alpha_i}$ from (21) we have $'\boldsymbol{s}^{\alpha_i'}=\sigma^{\alpha_i'}_{\alpha_i}\boldsymbol{s}^{\alpha_i}$. Therefore, in general case we obtain

$$'\boldsymbol{s}^{\alpha_1'\alpha_2'\cdots\alpha_k'}=\sum\sigma^{\alpha_1'}_{\alpha_1}\sigma^{\alpha_2'}_{\alpha_2}\cdots\sigma^{\alpha_k'}_{\alpha_k}\boldsymbol{s}^{\alpha_1\alpha_2\cdots\alpha_k}. \tag{22}$$

A representation (22) is called *undotted spintensor representation of the proper Lorentz group of the rank k.*

Further, let $\overset{*}{\mathbb{C}}_2$ be the Pauli algebra with the coefficients which are complex conjugate to the coefficients of $\mathbb{C}_2$. Let us show that the algebra $\overset{*}{\mathbb{C}}_2$ is derived from $\mathbb{C}_2$ under action of the automorphism $\mathcal{A}\rightarrow\mathcal{A}^\star$ or antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}}$. Indeed, in virtue of an isomorphism $\mathbb{C}_2\simeq C\!\ell_{3,0}$ a general element

$$\mathcal{A}=a^0\mathbf{e}_0+\sum_{i=1}^3 a^i\mathbf{e}_i+\sum_{i=1}^3\sum_{j=1}^3 a^{ij}\mathbf{e}_{ij}+a^{123}\mathbf{e}_{123}$$

of the algebra $C\!\ell_{3,0}$ can be written in the form

$$\mathcal{A}=(a^0+\omega a^{123})\mathbf{e}_0+(a^1+\omega a^{23})\mathbf{e}_1+(a^2+\omega a^{31})\mathbf{e}_2+(a^3+\omega a^{12})\mathbf{e}_3, \tag{23}$$

where $\omega=\mathbf{e}_{123}$. Since ω belongs to a center of the algebra $C\!\ell_{3,0}$ (ω commutes with all the basis elements) and $\omega^2=-1$, then we can to suppose $\omega\equiv i$. The action of the automorphism $\star$ on the homogeneous element $\mathcal{A}$ of the degree k is defined by the formula $\mathcal{A}^\star=(-1)^k\mathcal{A}$. In accordance with this the action of the automorphism $\mathcal{A}\rightarrow\mathcal{A}^\star$, where $\mathcal{A}$ is the element (23), has the form

$$\mathcal{A}\longrightarrow\mathcal{A}^\star=-(a^0-\omega a^{123})\mathbf{e}_0-(a^1-\omega a^{23})\mathbf{e}_1-(a^2-\omega a^{31})\mathbf{e}_2-(a^3-\omega a^{12})\mathbf{e}_3. \tag{24}$$

Therefore, $\star:\ \mathbb{C}_2 \rightarrow -\overset{*}{\mathbb{C}}_2$. Correspondingly, the action of the antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ on the homogeneous element $\mathcal{A}$ of the degree k is defined by the formula $\widetilde{\mathcal{A}} = (-1)^{\frac{k(k-1)}{2}}\mathcal{A}$. Thus, for the element (23) we obtain

$$\mathcal{A} \longrightarrow \widetilde{\mathcal{A}} = (a^0 - \omega a^{123})\mathbf{e}_0 + (a^1 - \omega a^{23})\mathbf{e}_1 + (a^2 - \omega a^{31})\mathbf{e}_2 + (a^3 - \omega a^{12})\mathbf{e}_3, \tag{25}$$

that is, $\widetilde{\ }:\ \mathbb{C}_2 \rightarrow \overset{*}{\mathbb{C}}_2$. This allows us to define an algebraic analogue of the Wigner's representation doubling: $\mathbb{C}_2 \oplus \overset{*}{\mathbb{C}}_2$. Further, from (23) it follows that $\mathcal{A} = \mathcal{A}_1 + \omega\mathcal{A}_2 = (a^0\mathbf{e}_0 + a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3) + \omega(a^{123}\mathbf{e}_0 + a^{23}\mathbf{e}_1 + a^{31}\mathbf{e}_2 + a^{12}\mathbf{e}_3)$. In general case, by virtue of an isomorphism $\mathbb{C}_{2k} \simeq C\!\ell_{p,q}$, where $C\!\ell_{p,q}$ is a real Clifford algebra with a division ring $\mathbb{K} \simeq \mathbb{C}$, $p - q \equiv 3, 7 \pmod 8$, we have for the general element of $C\!\ell_{p,q}$ an expression $\mathcal{A} = \mathcal{A}_1 + \omega\mathcal{A}_2$, here $\omega^2 = \mathbf{e}^2_{12\ldots p+q} = -1$ and, therefore, $\omega \equiv i$. Thus, from $\mathbb{C}_{2k}$ under action of the automorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$ we obtain a general algebraic doubling

$$\mathbb{C}_{2k} \oplus \overset{*}{\mathbb{C}}_{2k}. \tag{26}$$

The tensor product $\overset{*}{\mathbb{C}}_2 \otimes \overset{*}{\mathbb{C}}_2 \otimes \cdots \otimes \overset{*}{\mathbb{C}}_2 \simeq \overset{*}{\mathbb{C}}_{2r}$ of the r algebras $\overset{*}{\mathbb{C}}_2$ induces the tensor product of the r spinspaces $\dot{\mathbb{S}}_2$:

$$\dot{\mathbb{S}}_2 \otimes \dot{\mathbb{S}}_2 \otimes \cdots \otimes \dot{\mathbb{S}}_2 = \dot{\mathbb{S}}_{2^r}.$$

Vectors of the spinspace $\dot{\mathbb{S}}_{2^r}$ has the form

$$\boldsymbol{s}^{\dot{\alpha}_1\dot{\alpha}_2\cdots\dot{\alpha}_r} = \sum \boldsymbol{s}^{\dot{\alpha}_1} \otimes \boldsymbol{s}^{\dot{\alpha}_2} \otimes \cdots \otimes \boldsymbol{s}^{\dot{\alpha}_r}, \tag{27}$$

where the each cospinor $\boldsymbol{s}^{\dot{\alpha}_i}$ from (27) is transformed by the rule $'\boldsymbol{s}^{\dot{\alpha}'_i} = \sigma^{\dot{\alpha}'_i}_{\dot{\alpha}_i}\boldsymbol{s}^{\dot{\alpha}_i}$. Therefore,

$$'\boldsymbol{s}^{\dot{\alpha}'_1\dot{\alpha}'_2\cdots\dot{\alpha}'_r} = \sum \sigma^{\dot{\alpha}'_1}_{\dot{\alpha}_1}\sigma^{\dot{\alpha}'_2}_{\dot{\alpha}_2}\cdots\sigma^{\dot{\alpha}'_r}_{\dot{\alpha}_r}\boldsymbol{s}^{\dot{\alpha}_1\dot{\alpha}_2\cdots\dot{\alpha}_r}. \tag{28}$$

The representation (28) is called *a dotted spintensor representation of the proper Lorentz group of the rank r.*

In general case we have a tensor product of the k algebras $\mathbb{C}_2$ and the r algebras $\overset{*}{\mathbb{C}}_2$:

$$\mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2 \bigotimes \overset{*}{\mathbb{C}}_2 \otimes \overset{*}{\mathbb{C}}_2 \otimes \cdots \otimes \overset{*}{\mathbb{C}}_2 \simeq \mathbb{C}_{2k} \otimes \overset{*}{\mathbb{C}}_{2r},$$

which induces a spinspace

$$\mathbb{S}_2 \otimes \mathbb{S}_2 \otimes \cdots \otimes \mathbb{S}_2 \bigotimes \dot{\mathbb{S}}_2 \otimes \dot{\mathbb{S}}_2 \otimes \cdots \otimes \dot{\mathbb{S}}_2 = \mathbb{S}_{2^{k+r}}$$

with the vectors

$$\boldsymbol{s}^{\alpha_1\alpha_2\cdots\alpha_k\dot{\alpha}_1\dot{\alpha}_2\cdots\dot{\alpha}_r} = \sum \boldsymbol{s}^{\alpha_1} \otimes \boldsymbol{s}^{\alpha_2} \otimes \cdots \otimes \boldsymbol{s}^{\alpha_k} \otimes \boldsymbol{s}^{\dot{\alpha}_1} \otimes \boldsymbol{s}^{\dot{\alpha}_2} \otimes \cdots \otimes \boldsymbol{s}^{\dot{\alpha}_r}. \tag{29}$$

In this case we have a natural unification of the representations (22) and (28):

$$'\boldsymbol{s}^{\alpha'_1\alpha'_2\cdots\alpha'_k\dot{\alpha}'_1\dot{\alpha}'_2\cdots\dot{\alpha}'_r} = \sum \sigma^{\alpha'_1}_{\alpha_1}\sigma^{\alpha'_2}_{\alpha_2}\cdots\sigma^{\alpha'_k}_{\alpha_k}\sigma^{\dot{\alpha}'_1}_{\dot{\alpha}_1}\sigma^{\dot{\alpha}'_2}_{\dot{\alpha}_2}\cdots\sigma^{\dot{\alpha}'_r}_{\dot{\alpha}_r}\boldsymbol{s}^{\alpha_1\alpha_2\cdots\alpha_k\dot{\alpha}_1\dot{\alpha}_2\cdots\dot{\alpha}_r}. \tag{30}$$

So, a representation (30) is called *a spintensor representation of the proper Lorentz group of the rank (k, r).*

Further, let $\mathfrak{g} \rightarrow T_{\mathfrak{g}}$ be an arbitrary linear representation of the proper orthochronous Lorentz group $\mathfrak{G}_+ = \mathrm{SO}_0(1,3)$ and let $\mathsf{A}_i(t) = T_{a_i(t)}$ be an infinitesimal operator corresponding to the

rotation $a_i(t) \in \mathfrak{G}_+$. Analogously, let $\mathsf{B}_i(t) = T_{b_i(t)}$, where $b_i(t) \in \mathfrak{G}_+$ is the hyperbolic rotation. The operators A_i and B_i satisfy to the following relations:

$$\left.\begin{array}{lll}
[\mathsf{A}_1, \mathsf{A}_2] = \mathsf{A}_3, & [\mathsf{A}_2, \mathsf{A}_3] = \mathsf{A}_1, & [\mathsf{A}_3, \mathsf{A}_1] = \mathsf{A}_2, \\
[\mathsf{B}_1, \mathsf{B}_2] = -\mathsf{A}_3, & [\mathsf{B}_2, \mathsf{B}_3] = -\mathsf{A}_1, & [\mathsf{B}_3, \mathsf{B}_1] = -\mathsf{A}_2, \\
[\mathsf{A}_1, \mathsf{B}_1] = 0, & [\mathsf{A}_2, \mathsf{B}_2] = 0, & [\mathsf{A}_3, \mathsf{B}_3] = 0, \\
[\mathsf{A}_1, \mathsf{B}_2] = \mathsf{B}_3, & [\mathsf{A}_1, \mathsf{B}_3] = -\mathsf{B}_2, & \\
[\mathsf{A}_2, \mathsf{B}_3] = \mathsf{B}_1, & [\mathsf{A}_2, \mathsf{B}_1] = -\mathsf{B}_3, & \\
[\mathsf{A}_3, \mathsf{B}_1] = \mathsf{B}_2, & [\mathsf{A}_3, \mathsf{B}_2] = -\mathsf{B}_1. &
\end{array}\right\} \qquad (31)$$

Denoting $\mathsf{I}^{23} = \mathsf{A}_1$, $\mathsf{I}^{31} = \mathsf{A}_2$, $\mathsf{I}^{12} = \mathsf{A}_3$, and $\mathsf{I}^{01} = \mathsf{B}_1$, $\mathsf{I}^{02} = \mathsf{B}_2$, $\mathsf{I}^{03} = \mathsf{B}_3$ we write the relations (31) in a more compact form:

$$\left[\mathsf{I}^{\mu\nu}, \mathsf{I}^{\lambda\rho}\right] = \delta_{\mu\rho}\mathsf{I}^{\lambda\nu} + \delta_{\nu\lambda}\mathsf{I}^{\mu\rho} - \delta_{\nu\rho}\mathsf{I}^{\mu\lambda} - \delta_{\mu\lambda}\mathsf{I}^{\nu\rho}.$$

As is known [21], finite-dimensional (spinor) representations of the group $\mathrm{SO}_0(1,3)$ in the space of symmetrical polynomials $\mathrm{Sym}_{(k,r)}$ have the following form:

$$T_{\mathfrak{g}}q(\xi,\overline{\xi}) = (\gamma\xi+\delta)^{l_0+l_1-1}\overline{(\gamma\xi+\delta)}^{l_0-l_1+1} q\left(\frac{\alpha\xi+\beta}{\gamma\xi+\delta};\overline{\frac{\alpha\xi+\beta}{\gamma\xi+\delta}}\right), \qquad (32)$$

where $k = l_0 + l_1 - 1$, $r = l_0 - l_1 + 1$, and the pair (l_0, l_1) defines some representation of the group $\mathrm{SO}_0(1,3)$ in the Gel'fand-Naimark basis:

$$H_3\xi_{k\nu} = m\xi_{k\nu},$$

$$H_+\xi_{k\nu} = \sqrt{(k+\nu+1)(k-\nu)}\xi_{k,\nu+1},$$

$$H_-\xi_{k\nu} = \sqrt{(k+\nu)(k-\nu+1)}\xi_{k,\nu-1},$$

$$F_3\xi_{k\nu} = C_l\sqrt{k^2-\nu^2}\xi_{k-1,\nu} - A_l\nu\xi_{k,\nu} - C_{k+1}\sqrt{(k+1)^2-\nu^2}\xi_{k+1,\nu},$$

$$F_+\xi_{k\nu} = C_k\sqrt{(k-\nu)(k-\nu-1)}\xi_{k-1,\nu+1} - A_k\sqrt{(k-\nu)(k+\nu+1)}\xi_{k,\nu+1} + \\ + C_{k+1}\sqrt{(k+\nu+1)(k+\nu+2)}\xi_{k+1,\nu+1},$$

$$F_-\xi_{k\nu} = -C_k\sqrt{(k+\nu)(k+\nu-1)}\xi_{k-1,\nu-1} - A_k\sqrt{(k+\nu)(k-\nu+1)}\xi_{k,\nu-1} - \\ - C_{k+1}\sqrt{(k-\nu+1)(k-\nu+2)}\xi_{k+1,\nu-1},$$

$$A_k = \frac{\mathbf{i}l_0l_1}{k(k+1)}, \quad C_k = \frac{\mathbf{i}}{k}\sqrt{\frac{(k^2-l_0^2)(k^2-l_1^2)}{4k^2-1}}, \qquad (33)$$

$$\nu = -k, -k+1, \ldots, k-1, k,$$

$$k = l_0\,, l_0+1, \ldots,$$

where l_0 is positive integer or half-integer number, l_1 is an arbitrary complex number. These formulae define a finite–dimensional representation of the group $\mathrm{SO}_0(1,3)$ when $l_1^2 = (l_0+p)^2$, p is some natural number. In the case $l_1^2 \neq (l_0+p)^2$ we have an infinite-dimensional representation of $\mathrm{SO}_0(1,3)$. The operators $H_3, H_+, H_-, F_3, F_+, F_-$ are

$$H_+ = \mathbf{i}\mathsf{A}_1 - \mathsf{A}_2, \quad H_- = \mathbf{i}\mathsf{A}_1 + \mathsf{A}_2, \quad H_3 = \mathbf{i}\mathsf{A}_3,$$

$$F_+ = \mathbf{i}\mathsf{B}_1 - \mathsf{B}_2, \quad F_- = \mathbf{i}\mathsf{B}_1 + \mathsf{B}_2, \quad F_3 = \mathbf{i}\mathsf{B}_3.$$

Let us consider the operators

$$\mathsf{X}_l = \frac{1}{2}\mathbf{i}(\mathsf{A}_l + \mathbf{i}\mathsf{B}_l), \quad \mathsf{Y}_l = \frac{1}{2}\mathbf{i}(\mathsf{A}_l - \mathbf{i}\mathsf{B}_l), \qquad (34)$$
$$(l = 1, 2, 3).$$

Using the relations (31), we find that

$$[\mathsf{X}_k, \mathsf{X}_l] = \mathbf{i}\varepsilon_{klm}\mathsf{X}_m, \quad [\mathsf{Y}_l, \mathsf{Y}_m] = \mathbf{i}\varepsilon_{lmn}\mathsf{Y}_n, \quad [\mathsf{X}_l, \mathsf{Y}_m] = 0. \qquad (35)$$

Further, introducing generators of the form

$$\left.\begin{array}{l}\mathsf{X}_+ = \mathsf{X}_1 + \mathbf{i}\mathsf{X}_2, \quad \mathsf{X}_- = \mathsf{X}_1 - \mathbf{i}\mathsf{X}_2, \\ \mathsf{Y}_+ = \mathsf{Y}_1 + \mathbf{i}\mathsf{Y}_2, \quad \mathsf{Y}_- = \mathsf{Y}_1 - \mathbf{i}\mathsf{Y}_2,\end{array}\right\} \qquad (36)$$

we see that in virtue of commutativity of the relations (35) a space of an irreducible finite–dimensional representation of the group $\mathrm{SL}(2,\mathbb{C})$ can be spanned on the totality of $(2l+1)(2\dot{l}+1)$ basis vectors $\mid l, m; \dot{l}, \dot{m}\rangle$, where $l, m, \dot{l}, \dot{m}$ are integer or half–integer numbers, $-l \leq m \leq l$, $-\dot{l} \leq \dot{m} \leq \dot{l}$. Therefore,

$$\begin{array}{l}
\mathsf{X}_- \mid l, m; \dot{l}, \dot{m}\rangle = \sqrt{(l+m)(l-m+1)} \mid l, m-1, \dot{l}, \dot{m}\rangle \ \ (m > -l),\\
\mathsf{X}_+ \mid l, m; \dot{l}, \dot{m}\rangle = \sqrt{(l-m)(l+m+1)} \mid l, m+1; \dot{l}, \dot{m}\rangle \ \ (m < l),\\
\mathsf{X}_3 \mid l, m; \dot{l}, \dot{m}\rangle = m \mid l, m; \dot{l}, \dot{m}\rangle,\\
\mathsf{Y}_- \mid l, m; \dot{l}, \dot{m}\rangle = \sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}+1)} \mid l, m; \dot{l}, \dot{m}-1\rangle \ \ (\dot{m} > -\dot{l}),\\
\mathsf{Y}_+ \mid l, m; \dot{l}, \dot{m}\rangle = \sqrt{(\dot{l}-\dot{m})(\dot{l}+\dot{m}+1)} \mid l, m; \dot{l}, \dot{m}+1\rangle \ \ (\dot{m} < \dot{l}),\\
\mathsf{Y}_3 \mid l, m; \dot{l}, \dot{m}\rangle = \dot{m} \mid l, m; \dot{l}, \dot{m}\rangle.
\end{array} \qquad (37)$$

From the relations (35) it follows that each of the sets of infinitesimal operators X and Y generates the group $\mathrm{SU}(2)$ and these two groups commute with each other. Thus, from the relations (35) and (37) it follows that the group $\mathrm{SL}(2,\mathbb{C})$, in essence, is equivalent locally to the group $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$. In contrast to the Gel'fand–Naimark representation for the Lorentz group [21, 36], which does not find a broad application in physics, a representation (37) is a most useful in theoretical physics (see, for example, [1, 44, 41, 42]). This representation for the Lorentz group was first given by Van der Waerden in [62]. It should be noted here that the representation basis, defined by the formulae (34)–(37), has an evident physical meaning. For example, in the case of $(1,0)\oplus(0,1)$–representation space there is an analogy with the photon spin states. Namely, the operators X and Y correspond to the right and left polarization states of the photon. The following relations between generators $\mathsf{Y}_\pm$, $\mathsf{X}_\pm$, Y_3, X_3 and $H_\pm$, $F_\pm$, H_3, F_3 define a relationship between the Van der Waerden and Gel'fand-Naimark bases:

$$\begin{array}{rclrcl}
\mathsf{Y}_+ & = & -\frac{1}{2}(F_+ + \mathbf{i}H_+), & \mathsf{X}_+ & = & \frac{1}{2}(F_+ - \mathbf{i}H_+),\\
\mathsf{Y}_- & = & -\frac{1}{2}(F_- + \mathbf{i}H_-), & \mathsf{X}_- & = & \frac{1}{2}(F_- - \mathbf{i}H_-),\\
\mathsf{Y}_3 & = & -\frac{1}{2}(F_3 + \mathbf{i}H_3), & \mathsf{X}_3 & = & \frac{1}{2}(F_3 - \mathbf{i}H_3).
\end{array}$$

The relation between the numbers l_0, l_1 and the number l (the weight of representation in the basis (37)) is given by the following formula:

$$(l_0, l_1) = (l, l+1)\,.$$

Whence it immediately follows that

$$l = \frac{l_0 + l_1 - 1}{2}. \tag{38}$$

As is known [21], if an irreducible representation of the proper Lorentz group $\mathrm{SO}_0(1,3)$ is defined by the pair (l_0, l_1), then a conjugated representation is also irreducible and is defined by a pair $\pm(l_0, -l_1)$. Therefore,

$$(l_0, l_1) = \left(-\dot{l},\, \dot{l} + 1\right).$$

Thus,

$$\dot{l} = \frac{l_0 - l_1 + 1}{2}. \tag{39}$$

Further, representations $\boldsymbol{\tau}_{s_1,s_2}$ and $\boldsymbol{\tau}_{s_1',s_2'}$ are called *interlocking irreducible representations of the Lorentz group* , that is, such representations that $s_1' = s_1 \pm \frac{1}{2}$, $s_2' = s_2 \pm \frac{1}{2}$ [20]. The two most full schemes of the interlocking irreducible representations of the Lorentz group (Gel'fand-Yaglom chains) for integer and half-integer spins are shown on the Fig. 1 and Fig. 2. As follows from

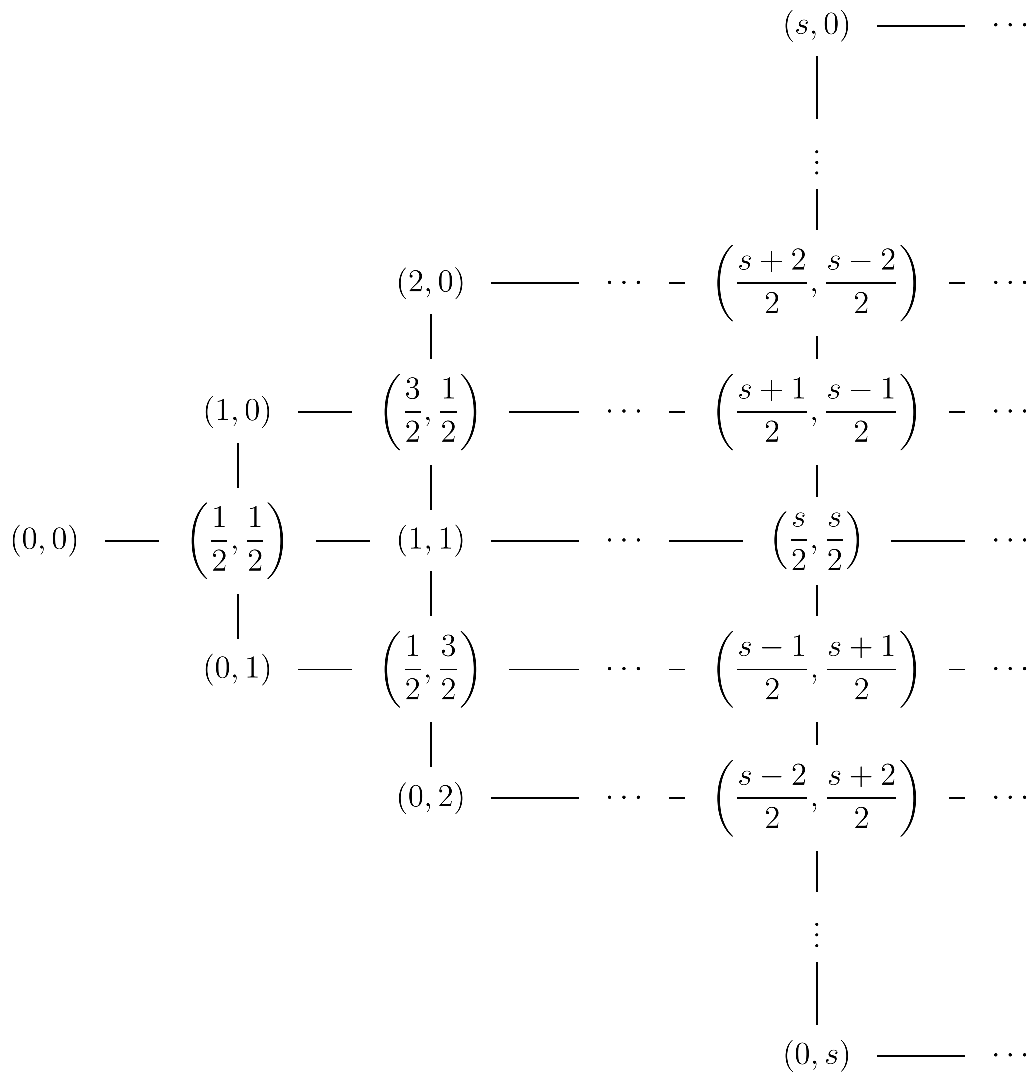

Fig. 1: Interlocking representation scheme for the fields of integer spin (Bose-scheme).

Fig. 1 the simplest field is the scalar field

$$(0,0).$$

$$
\begin{array}{ccccccccc}
 & & & & & & (s,0) & \text{———} & \cdots \\
 & & & & & & \Big| & & \\
 & & & & & & \vdots & & \\
 & & & & & & \Big| & & \\
 & & \left(\frac{3}{2},0\right) & \text{———} & \cdots & - & \left(\frac{2s+3}{4},\frac{2s-3}{4}\right) & - & \cdots \\
 & & | & & & & | & & \\
\left(\frac{1}{2},0\right) & \text{—} & \left(1,\frac{1}{2}\right) & \text{———} & \cdots & - & \left(\frac{2s+1}{4},\frac{2s-1}{4}\right) & - & \cdots \\
| & & | & & & & | & & \\
\left(0,\frac{1}{2}\right) & \text{—} & \left(\frac{1}{2},1\right) & \text{———} & \cdots & - & \left(\frac{2s-1}{4},\frac{2s+1}{4}\right) & - & \cdots \\
 & & | & & & & | & & \\
 & & \left(0,\frac{3}{2}\right) & \text{———} & \cdots & - & \left(\frac{2s-3}{4},\frac{2s+3}{4}\right) & - & \cdots \\
 & & & & & & \Big| & & \\
 & & & & & & \vdots & & \\
 & & & & & & \Big| & & \\
 & & & & & & (0,s) & \text{———} & \cdots
\end{array}
$$

Fig. 2: Interlocking representation scheme for the fields of half-integer spin (Fermi-scheme).

This field is described by the Fock-Klein-Gordon equation. In its turn, the simplest field from the Fermi-scheme (Fig. 2) is the electron-positron (spinor) field corresponding to the following interlocking scheme:

$$\left(\frac{1}{2},0\right) \longleftrightarrow \left(0,\frac{1}{2}\right) .$$

This field is described by the Dirac equation. Further, the next field from the Bose-scheme (Fig. 1) is a photon field (Maxwell field) defined within the interlocking scheme

$$(1,0) \longleftrightarrow \left(\frac{1}{2},\frac{1}{2}\right) \longleftrightarrow (0,1) .$$

This interlocking scheme leads to the Maxwell equations. The fields $(1/2,0)\oplus(0,1/2)$ and $(1,0)\oplus(0,1)$ (Dirac and Maxwell fields) are particular cases of fields of the type $(l,0)\oplus(0,l)$. Wave equations for such fields and their general solutions were found in the works [54, 56, 59].

It is easy to see that the interlocking scheme, corresponded to the Maxwell field, contains the field of tensor type:

$$\left(\frac{1}{2},\frac{1}{2}\right).$$

Further, the next interlocking scheme (see Fig. 2)

$$\left(\frac{3}{2},0\right)\longleftrightarrow\left(1,\frac{1}{2}\right)\longleftrightarrow\left(\frac{1}{2},1\right)\longleftrightarrow\left(0,\frac{3}{2}\right)\,,$$

corresponding to the Pauli-Fierz equations [19], contains a chain of the type

$$\left(1,\frac{1}{2}\right)\longleftrightarrow\left(\frac{1}{2},1\right)\,.$$

In such a way we come to wave equations for the fields $\boldsymbol{\psi}(\boldsymbol{\alpha})=\langle x,\mathfrak{g}\,|\boldsymbol{\psi}\rangle$ of tensor type $(l_1,l_2)\oplus(l_2,l_1)$. Wave equations for such fields and their general solutions were found in the work [61].

A relation between the numbers l_0, l_1 of the Gel'fand-Naimark representation (33) and the number k of the factors $\mathbb{C}_2$ in the product $\mathbb{C}_2\otimes\mathbb{C}_2\otimes\cdots\otimes\mathbb{C}_2$ is given by the following formula:

$$(l_0,l_1)=\left(\frac{k}{2},\frac{k}{2}+1\right),$$

Hence it immediately follows that $k=l_0+l_1-1$. Thus, we have *a complex representation* $\mathfrak{C}^{l_0+l_1-1,0}$ *of the group* $\mathbf{Spin}_+(1,3)$ *in the spinspace* $\mathbb{S}_{2^k}$. If the representation $\mathfrak{C}^{l_0+l_1-1,0}$ is reducible, then the space $\mathbb{S}_{2^k}$ is decomposed into a direct sum of irreducible subspaces, that is, it is possible to choose in $\mathbb{S}_{2^k}$ such a basis, in which all the matrices take a block-diagonal form. Then the field $\boldsymbol{\psi}(\boldsymbol{\alpha})$ is reduced to some number of the fields corresponding to irreducible representations of the group $\mathbf{Spin}_+(1,3)$, each of which is transformed independently from the other, and the field $\boldsymbol{\psi}(\boldsymbol{\alpha})$ in this case is a collection of the fields with more simple structure. It is obvious that these more simple fields correspond to irreducible representations $\mathfrak{C}$.

Analogously, a relation between the numbers l_0, l_1 of the Gel'fand-Naimark representation (33) and the number r of the factors $\overset{*}{\mathbb{C}}_2$ in the product $\overset{*}{\mathbb{C}}_2\otimes\overset{*}{\mathbb{C}}_2\otimes\cdots\otimes\overset{*}{\mathbb{C}}_2$ is given by the following formula:

$$(-l_0,l_1)=\left(-\frac{r}{2},\frac{r}{2}+1\right).$$

Hence it immediately follows that $r=l_0-l_1+1$. Thus, we have a complex representation $\mathfrak{C}^{0,l_0-l_1+1}$ of $\mathbf{Spin}_+(1,3)$ in the spinspace $\mathbb{S}_{2^r}$.

As is known [36, 21, 41], a system of irreducible finite-dimensional representations of the group $\mathfrak{G}_+$ is realized in the space $\mathrm{Sym}_{(k,r)}\subset\mathbb{S}_{2^{k+r}}$ of symmetric spintensors. The dimensionality of $\mathrm{Sym}_{(k,r)}$ is equal to $(k+1)(r+1)$. A representation of the group $\mathfrak{G}_+$, defined by such spintensors, is irreducible and denoted by the symbol $\mathfrak{D}^{(l,\dot{l})}(\sigma)$, where $2l=k$, $2\dot{l}=r$, the numbers l and $\dot{l}$ are integer or half-integer. In general case, the field $\boldsymbol{\psi}(\boldsymbol{\alpha})$ is the field of type $(l,\dot{l})$. As a rule, in physics there are two basic types of the fields:

1) The field of type $(l,0)$. The structure of this field (or the field $(0,\dot{l})$) is described by the representation $\mathfrak{D}^{(l,0)}(\sigma)$ ($\mathfrak{D}^{(0,\dot{l})}(\sigma)$), which is realized in the space $\mathbb{S}_{2^k}$ ($\mathbb{S}_{2^r}$). At this point, the algebra $\mathbb{C}_{2k}\simeq\mathbb{C}_2\otimes\mathbb{C}_2\otimes\cdots\otimes\mathbb{C}_2$ (correspondingly, $\overset{*}{\mathbb{C}}_{2k}\simeq\overset{*}{\mathbb{C}}_2\otimes\overset{*}{\mathbb{C}}_2\otimes\cdots\otimes\overset{*}{\mathbb{C}}_2$) is associated with the field of the type $(l,0)$ (correspondingly, $(0,\dot{l})$). The trivial case $l=0$ corresponds to *a Pauli-Weisskopf field* describing the scalar particles. Further, at $l=\dot{l}=1/2$ we have *a Weyl field* describing the neutrino. At this point the antineutrino is described by a fundamental representation $\mathfrak{D}^{(1/2,0)}(\sigma)=\sigma$ of the group $\mathfrak{G}_+$ and the algebra $\mathbb{C}_2$. Correspondingly, the neutrino is described by a conjugated representation $\mathfrak{D}^{(0,1/2)}(\sigma)$ and the algebra $\overset{*}{\mathbb{C}}_2$. In essence, one can

say that the algebra $\mathbb{C}_2$ ($\overset{*}{\mathbb{C}}_2$) is the basic building block, from which other physical fields built by means of direct sum or tensor product. One can say that this situation looks like the de Broglie fusion method [8]

2) The field of type $(l,0)\oplus(0,\dot{l})$. The structure of this field admits a space inversion and, therefore, in accordance with a Wigner's doubling [63] is described by a representation $\mathfrak{D}^{(l,0)}\oplus\mathfrak{D}^{(0,\dot{l})}$ of the group $\mathfrak{G}_+$. This representation is realized in the space $\mathbb{S}_{2^{2k}}$. The Clifford algebra, related with this representation, is a direct sum $\mathbb{C}_{2k}\oplus\overset{*}{\mathbb{C}}_{2k}\simeq\mathbb{C}_2\otimes\mathbb{C}_2\otimes\cdots\otimes\mathbb{C}_2\bigoplus\overset{*}{\mathbb{C}}_2\otimes\overset{*}{\mathbb{C}}_2\otimes\cdots\otimes\overset{*}{\mathbb{C}}_2$. In the simplest case $l=1/2$ we have *bispinor (electron–positron) Dirac field* $(1/2,0)\oplus(0,1/2)$ with the algebra $\mathbb{C}_2\oplus\overset{*}{\mathbb{C}}_2$. It should be noted that the Dirac algebra $\mathbb{C}_4$, considered as a tensor product $\mathbb{C}_2\otimes\mathbb{C}_2$ (or $\mathbb{C}_2\otimes\overset{*}{\mathbb{C}}_2$) in accordance with (21) (or (29)) gives rise to spintensors $\boldsymbol{s}^{\alpha_1\alpha_2}$ (or $\boldsymbol{s}^{\alpha_1\dot{\alpha}_1}$), but it contradicts with the usual definition of the Dirac bispinor as a pair $(\boldsymbol{s}^{\alpha_1},\boldsymbol{s}^{\dot{\alpha}_1})$. Therefore, the Clifford algebra, associated with the Dirac field, is $\mathbb{C}_2\oplus\overset{*}{\mathbb{C}}_2$, and a spinspace of this sum in virtue of unique decomposition $\mathbb{S}_2\oplus\dot{\mathbb{S}}_2=\mathbb{S}_4$ is a spinspace of $\mathbb{C}_4$.

Spinor representations of the units of $\mathbb{C}_n$ we will define in the Brauer-Weyl representation [7]:

$$
\begin{array}{lcl}
\mathcal{E}_1 & = & \sigma_1\otimes\mathbf{1}_2\otimes\cdots\otimes\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2,\\
\mathcal{E}_2 & = & \sigma_3\otimes\sigma_1\otimes\mathbf{1}_2\otimes\cdots\otimes\mathbf{1}_2\otimes\mathbf{1}_2,\\
\mathcal{E}_3 & = & \sigma_3\otimes\sigma_3\otimes\sigma_1\otimes\mathbf{1}_2\otimes\cdots\otimes\mathbf{1}_2,\\
\multicolumn{3}{l}{\dotfill}\\
\mathcal{E}_m & = & \sigma_3\otimes\sigma_3\otimes\cdots\otimes\sigma_3\otimes\sigma_1,\\
\mathcal{E}_{m+1} & = & \sigma_2\otimes\mathbf{1}_2\otimes\cdots\otimes\mathbf{1}_2,\\
\mathcal{E}_{m+2} & = & \sigma_3\otimes\sigma_2\otimes\mathbf{1}_2\otimes\cdots\otimes\mathbf{1}_2,\\
\multicolumn{3}{l}{\dotfill}\\
\mathcal{E}_{2m} & = & \sigma_3\otimes\sigma_3\otimes\cdots\otimes\sigma_3\otimes\sigma_2,
\end{array}
\tag{40}
$$

where

$$
\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad\sigma_3=\begin{pmatrix}i&0\\0&-i\end{pmatrix}
$$

are spinor representations of the units of $\mathbb{C}_2$, $\mathbf{1}_2$ is the unit 2×2 matrix.

3) Tensor fields $(l',l'')\oplus(\dot{l}'',\dot{l}')$. The fields (l',l'') and $(\dot{l}'',\dot{l}')$ are defined within the arbitrary spin chains (see Fig. 1 and Fig. 2). Universal coverings of these spin chains are constructed within the representations $\mathfrak{C}^{l_0+l_1-1,l_0-l_1+1}$ and $\mathfrak{C}^{l_0-l_1+1,l_0+l_1-1}$ of $\mathbf{Spin}_+(1,3)$ in the spinspaces $\mathbb{S}_{2^{k+r}}$ associated with the algebra $\mathbb{C}_2\otimes\mathbb{C}_2\otimes\cdots\otimes\mathbb{C}_2\bigotimes\overset{*}{\mathbb{C}}_2\otimes\overset{*}{\mathbb{C}}_2\otimes\cdots\otimes\overset{*}{\mathbb{C}}_2$. A relation between the numbers l_0, l_1 of the Gel'fand-Naimark basis (33) and the numbers k and r of the factors $\mathbb{C}_2$ and $\overset{*}{\mathbb{C}}_2$ is given by the following formula:

$$
(l_0,l_1)=\left(\frac{k-r}{2},\frac{k+r}{2}+1\right).
$$

Finally, extended automorphisms groups $\operatorname{Ext}(\mathbb{C}_{2k}\oplus\overset{*}{\mathbb{C}}_{2k})$ and $\operatorname{Ext}(\mathbb{C}_{2k}\otimes\overset{*}{\mathbb{C}}_{2k})$ (correspondingly, CPT groups) can be derived via the same procedure that described in detail in our previous work [55]. $\square$

4 The CPT group of the spin-$1/2$ field

In accordance with the general Fermi-scheme (Fig. 1) of the interlocking representations of $\mathfrak{G}_+$ the field $(1/2,0)\oplus(0,1/2)$ is defined within the following chain:

$$\left(\frac{1}{2},0\right)\longleftrightarrow\left(0,\frac{1}{2}\right).$$

A double covering of the representation associated with the field $(1/2,0)\oplus(0,1/2)$ is realized in the spinspace $\mathbb{S}_2\oplus\dot{\mathbb{S}}_2$. This spinspace is a space of the representation $\mathfrak{C}^{1,0}\oplus\mathfrak{C}^{0,-1}$ of $\mathbf{Spin}_+(1,3)$. Further, the algebra $\mathbb{C}_2\oplus\overset{*}{\mathbb{C}}_2$ corresponds to $\mathfrak{C}^{1,0}\oplus\mathfrak{C}^{0,-1}$ and the automorphisms of this algebra are realized within the representations of $\mathbf{Pin}(1,3)$, that is, they are outer automorphisms with respect to the transformations of the group $\mathbf{Spin}_+(1,3)$. The spinor representations of the automorphisms, defined on the spinspace $\mathbb{S}_2\oplus\dot{\mathbb{S}}_2$, are constructed via the Brauer-Weyl representation (40). The spinbasis of the algebra $\mathbb{C}_2\oplus\overset{*}{\mathbb{C}}_2$ is defined by the following 4×4 matrices:

$$\begin{gathered}\mathcal{E}_1=\sigma_1\otimes\mathbf{1}_2=\begin{pmatrix}0 & \mathbf{1}_2\\ \mathbf{1}_2 & 0\end{pmatrix},\quad \mathcal{E}_2=\sigma_3\otimes\sigma_1=\begin{pmatrix}i\sigma_1 & 0\\ 0 & -i\sigma_1\end{pmatrix},\\ \mathcal{E}_3=\sigma_2\otimes\mathbf{1}_2=\begin{pmatrix}0 & -i\mathbf{1}_2\\ i\mathbf{1}_2 & \end{pmatrix},\mathcal{E}_4=\sigma_3\otimes\sigma_2=\begin{pmatrix}i\sigma_2 & 0\\ 0 & -i\sigma_2\end{pmatrix}.\end{gathered}\tag{41}$$

In accordance with (7) we have for the matrix of the automorphism $\mathcal{A}\rightarrow\mathcal{A}^\star$ the following expression:

$$\mathsf{W}=\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3\mathcal{E}_4=\mathcal{E}_{1234}\sim P.$$

Further, it is easy to see that among the matrices of the basis (41) there are symmetric and skewsymmetric matrices:

$$\mathcal{E}_1^{\mathsf{T}}=\mathcal{E}_1,\quad \mathcal{E}_2^{\mathsf{T}}=\mathcal{E}_2,\quad \mathcal{E}_3^{\mathsf{T}}=-\mathcal{E}_3,\quad \mathcal{E}_4^{\mathsf{T}}=-\mathcal{E}_4.$$

In accordance with $\widetilde{\mathsf{A}}=\mathsf{E}\mathsf{A}^{\mathsf{T}}\mathsf{E}^{-1}$ (see (8)) we have

$$\mathcal{E}_1=\mathsf{E}\mathcal{E}_1\mathsf{E}^{-1},\quad \mathcal{E}_2=\mathsf{E}\mathcal{E}_2\mathsf{E}^{-1},\quad \mathcal{E}_3=-\mathsf{E}\mathcal{E}_3\mathsf{E}^{-1},\quad \mathcal{E}_4=-\mathsf{E}\mathcal{E}_4\mathsf{E}^{-1}.$$

Hence it follows that E commutes with $\mathcal{E}_1$ and $\mathcal{E}_2$ and anticommutes with $\mathcal{E}_3$ and $\mathcal{E}_4$, that is, $\mathsf{E}=\mathcal{E}_3\mathcal{E}_4\sim T$. From the definition $\mathsf{C}=\mathsf{E}\mathsf{W}$ (see (9)) we find that the matrix of the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}^\star}$ has the form $\mathsf{C}=\mathcal{E}_1\mathcal{E}_2\sim PT$. The basis (41) contains both complex and real matrices:

$$\mathcal{E}_1^*=\mathcal{E}_1,\quad \mathcal{E}_2^*=-\mathcal{E}_2,\quad \mathcal{E}_3^*=-\mathcal{E}_3,\quad \mathcal{E}_4^*=\mathcal{E}_4.$$

Therefore, from $\overline{\mathsf{A}}=\Pi\mathsf{A}^*\Pi^{-1}$ (see (10)) we have

$$\mathcal{E}_1=\Pi\mathcal{E}_1\Pi^{-1},\quad \mathcal{E}_2=-\Pi\mathcal{E}_2\Pi^{-1},\quad \mathcal{E}_3=-\Pi\mathcal{E}_3\Pi^{-1},\quad \mathcal{E}_4=\Pi\mathcal{E}_4\Pi^{-1}.$$

From the latter relations we obtain $\Pi=\mathcal{E}_2\mathcal{E}_3\sim C$. Further, in accordance with $\mathsf{K}=\Pi\mathsf{W}$ (the definition (11)) for the matrix of the pseudoautomorphism $\mathcal{A}\rightarrow\overline{\mathcal{A}^\star}$ we have $\mathsf{K}=\mathcal{E}_1\mathcal{E}_4\sim CP$. Finally, for the pseudoantiautomorphisms $\mathcal{A}\rightarrow\overline{\widetilde{\mathcal{A}}}$ and $\mathcal{A}\rightarrow\overline{\widetilde{\mathcal{A}^\star}}$ from the definitions $\mathsf{S}=\Pi\mathsf{E}$ and $\mathsf{F}=\Pi\mathsf{C}$ (see (12) and (13)) it follows that $\mathsf{S}=\mathcal{E}_2\mathcal{E}_3\mathcal{E}_3\mathcal{E}_4=\mathcal{E}_2\mathcal{E}_4\sim CT$ and $\mathsf{F}=\mathcal{E}_2\mathcal{E}_3\mathcal{E}_1\mathcal{E}_2=\mathcal{E}_1\mathcal{E}_3\sim CPT$. Thus, we come to the following automorphism group:

$$\begin{aligned}\mathrm{Ext}(\mathbb{C}_4)=\{\mathsf{I},\mathsf{W},\mathsf{E},\mathsf{C},\Pi,\mathsf{K},\mathsf{S},\mathsf{F}\}\simeq\{1,P,T,PT,C,CP,CT,CPT\}\simeq\\ \{\mathbf{1}_4,\,\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3\mathcal{E}_4,\,\mathcal{E}_3\mathcal{E}_4,\,\mathcal{E}_1\mathcal{E}_2,\,\mathcal{E}_2\mathcal{E}_3,\,\mathcal{E}_1\mathcal{E}_4,\,\mathcal{E}_2\mathcal{E}_4,\,\mathcal{E}_1\mathcal{E}_4\}.\end{aligned}$$

The multiplication table of this group is shown in Tab. 4. From this table it follows that $\mathrm{Ext}(\mathbb{C}_4)\simeq D_4$, and for the CPT group we have the following isomorphism: $C^{+,+,+,+,+,-,-}\simeq D_4\otimes\mathbb{Z}_2$.

	$\mathbf{1}_4$	$\mathcal{E}_{1234}$	$\mathcal{E}_{34}$	$\mathcal{E}_{12}$	$\mathcal{E}_{23}$	$\mathcal{E}_{14}$	$\mathcal{E}_{24}$	$\mathcal{E}_{13}$
$\mathbf{1}_4$	$\mathbf{1}_4$	$\mathcal{E}_{1234}$	$\mathcal{E}_{34}$	$\mathcal{E}_{12}$	$\mathcal{E}_{23}$	$\mathcal{E}_{14}$	$\mathcal{E}_{24}$	$\mathcal{E}_{13}$
$\mathcal{E}_{1234}$	$\mathcal{E}_{1234}$	$\mathbf{1}_4$	$\mathcal{E}_{12}$	$\mathcal{E}_{34}$	$\mathcal{E}_{14}$	$\mathcal{E}_{23}$	$\mathcal{E}_{13}$	$\mathcal{E}_{24}$
$\mathcal{E}_{34}$	$\mathcal{E}_{34}$	$-\mathcal{E}_{12}$	$\mathbf{1}_4$	$\mathcal{E}_{1234}$	$-\mathcal{E}_{24}$	$-\mathcal{E}_{13}$	$-\mathcal{E}_{23}$	$-\mathcal{E}_{14}$
$\mathcal{E}_{12}$	$\mathcal{E}_{12}$	$\mathcal{E}_{34}$	$\mathcal{E}_{1234}$	$\mathbf{1}_4$	$-\mathcal{E}_{13}$	$-\mathcal{E}_{24}$	$-\mathcal{E}_{14}$	$-\mathcal{E}_{23}$
$\mathcal{E}_{23}$	$\mathcal{E}_{23}$	$\mathcal{E}_{14}$	$\mathcal{E}_{24}$	$\mathcal{E}_{13}$	$\mathbf{1}_4$	$\mathcal{E}_{1234}$	$\mathcal{E}_{34}$	$\mathcal{E}_{12}$
$\mathcal{E}_{14}$	$\mathcal{E}_{14}$	$\mathcal{E}_{23}$	$\mathcal{E}_{13}$	$\mathcal{E}_{24}$	$\mathcal{E}_{1234}$	$\mathbf{1}_4$	$\mathcal{E}_{12}$	$\mathcal{E}_{34}$
$\mathcal{E}_{24}$	$\mathcal{E}_{24}$	$\mathcal{E}_{13}$	$\mathcal{E}_{23}$	$\mathcal{E}_{14}$	$-\mathcal{E}_{34}$	$-\mathcal{E}_{12}$	$-\mathbf{1}_4$	$-\mathcal{E}_{1234}$
$\mathcal{E}_{13}$	$\mathcal{E}_{13}$	$\mathcal{E}_{24}$	$\mathcal{E}_{14}$	$\mathcal{E}_{23}$	$-\mathcal{E}_{12}$	$-\mathcal{E}_{34}$	$-\mathcal{E}_{1234}$	$-\mathbf{1}_4$

Tab. 4: The multiplication table of the $CPT/\mathbb{Z}_2$ group of the field $(1/2,0)\oplus(0,1/2)$.

5 The CPT group of the spin-1 field

In accordance with the general Bose-scheme of the interlocking representations of $\mathfrak{G}_+$ (see Fig. 1), the field $(1,0)\oplus(0,1)$ is defined within the following interlocking scheme:

$$(1,0) \longleftrightarrow \left(\frac{1}{2},\frac{1}{2}\right) \longleftrightarrow (0,1)\ .$$

A double covering of the representation, associated with the field $(1,0)\oplus(0,1)$, is realized in the spinspace

$$\mathbb{S}_2\otimes\mathbb{S}_2\bigoplus\dot{\mathbb{S}}_2\otimes\dot{\mathbb{S}}_2, \tag{42}$$

This spinspace is a space of the representation $\mathfrak{C}^{2,0}\oplus\mathfrak{C}^{0,-2}$ of the group $\mathbf{Spin}_+(1,3)$. The algebra

$$\mathbb{C}_2\otimes\mathbb{C}_2\bigoplus\overset{*}{\mathbb{C}}_2\otimes\overset{*}{\mathbb{C}}_2. \tag{43}$$

is associated with $\mathfrak{C}^{2,0}\oplus\mathfrak{C}^{0,-2}$. The automorphisms of this algebra are realized within representations of the group $\mathbf{Pin}(1,3)$, that is, they are outer automorphisms with respect transformations of the group $\mathbf{Spin}_+(1,3)$. Spinor representations of the automorphisms, defined on the spinspace (42), are constructed via the Brauer-Weyl representation (40). A spinbasis of the algebra (43) is defined by the following 8×8 matrices:

$$\mathcal{E}_1=\sigma_1\otimes\mathbf{1}_2\otimes\mathbf{1}_2=\begin{bmatrix} 0 & 0 & \mathbf{1}_2 & 0\\ 0 & 0 & 0 & \mathbf{1}_2\\ \mathbf{1}_2 & 0 & 0 & 0\\ 0 & \mathbf{1}_2 & 0 & 0\end{bmatrix},$$

$$\mathcal{E}_2=\sigma_3\otimes\sigma_1\otimes\mathbf{1}_2=\begin{bmatrix} 0 & i\mathbf{1}_1 & 0 & 0\\ i\mathbf{1}_2 & 0 & 0 & 0\\ 0 & 0 & 0 & -i\mathbf{1}_2\\ 0 & 0 & -i\mathbf{1}_2 & 0\end{bmatrix},$$

$$\mathcal{E}_3=\sigma_3\otimes\sigma_3\otimes\sigma_1=\begin{bmatrix}-\sigma_1 & 0 & 0 & 0\\ 0 & \sigma_1 & 0 & 0\\ 0 & 0 & \sigma_1 & 0\\ 0 & 0 & 0 & -\sigma_1\end{bmatrix},$$

$$\mathcal{E}_4=\sigma_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2=\begin{bmatrix}0 & 0 & -i\mathbf{1}_2 & 0\\ 0 & 0 & 0 & -i\mathbf{1}_2\\ i\mathbf{1}_2 & 0 & 0 & 0\\ 0 & i\mathbf{1}_2 & 0 & 0\end{bmatrix},$$

$$\mathcal{E}_5=\sigma_3\otimes\sigma_2\otimes\mathbf{1}_2=\begin{bmatrix}0 & \mathbf{1}_2 & 0 & 0\\ -\mathbf{1}_2 & 0 & 0 & 0\\ 0 & 0 & 0 & -\mathbf{1}_2\\ 0 & 0 & \mathbf{1}_2 & 0\end{bmatrix},$$

$$\mathcal{E}_6=\sigma_3\otimes\sigma_3\otimes\sigma_2=\begin{bmatrix}-\sigma_2 & 0 & 0 & 0\\ 0 & \sigma_2 & 0 & 0\\ 0 & 0 & \sigma_2 & 0\\ 0 & 0 & 0 & -\sigma_2\end{bmatrix}.$$

Using these matrices, we construct CPT group for the field $(1,0)\oplus(0,1)$. At first, the matrix of the automorphism $\mathcal{A}\rightarrow\mathcal{A}^\star$ has the form

$$\mathsf{W}=\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3\mathcal{E}_4\mathcal{E}_5\mathcal{E}_6=\mathcal{E}_{123456}\sim P.$$

Further, since

$$\mathcal{E}_1^{\mathsf{T}}=\mathcal{E}_1,\quad \mathcal{E}_2^{\mathsf{T}}=\mathcal{E}_2,\quad \mathcal{E}_3^{\mathsf{T}}=\mathcal{E}_3,\quad \mathcal{E}_4^{\mathsf{T}}=-\mathcal{E}_4,\quad \mathcal{E}_5^{\mathsf{T}}=-\mathcal{E}_5,\quad \mathcal{E}_6^{\mathsf{T}}=-\mathcal{E}_6,$$

then in accordance with $\widetilde{\mathsf{A}}=\mathsf{E}\mathsf{A}^{\mathsf{T}}\mathsf{E}^{-1}$ we have

$$\mathcal{E}_1=\mathsf{E}\mathcal{E}_1\mathsf{E}^{-1},\quad \mathcal{E}_2=\mathsf{E}\mathcal{E}_2\mathsf{E}^{-1},\quad \mathcal{E}_3=\mathsf{E}\mathcal{E}_3\mathsf{E}^{-1},\quad \mathcal{E}_4=-\mathsf{E}\mathcal{E}_4\mathsf{E}^{-1},$$
$$\mathcal{E}_5=-\mathsf{E}\mathcal{E}_5\mathsf{E}^{-1},\quad \mathcal{E}_6=-\mathsf{E}\mathcal{E}_6\mathsf{E}^{-1}.$$

Hence it follows that E commutes with $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ and anticommutes with $\mathcal{E}_4$, $\mathcal{E}_5$, $\mathcal{E}_6$, that is, $\mathsf{E}=\mathcal{E}_{456}\sim T$. From the definition $\mathsf{C}=\mathsf{E}\mathsf{W}$ we find that a matrix of the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}^\star}$ has the form $\mathsf{C}=\mathcal{E}_{123}\sim PT$. The basis $\{\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3,\mathcal{E}_4,\mathcal{E}_5,\mathcal{E}_6\}$ contains both complex and real matrices:

$$\mathcal{E}_1^*=\mathcal{E}_1,\quad \mathcal{E}_2^*=-\mathcal{E}_2,\quad \mathcal{E}_3^*=\mathcal{E}_3,\quad \mathcal{E}_4^*=-\mathcal{E}_4,\quad \mathcal{E}_5^*=\mathcal{E}_5,\quad \mathcal{E}_6^*=-\mathcal{E}_6.$$

Therefore, from $\overline{\mathsf{A}}=\Pi\mathsf{A}^*\Pi^{-1}$ we have

$$\mathcal{E}_1=\Pi\mathcal{E}_1\Pi^{-1},\quad \mathcal{E}_2=-\Pi\mathcal{E}_2\Pi^{-1},\quad \mathcal{E}_3=\Pi\mathcal{E}_3\Pi^{-1},\quad \mathcal{E}_4=-\Pi\mathcal{E}_4\Pi^{-1},$$
$$\mathcal{E}_5=\Pi\mathcal{E}_5\Pi^{-1},\quad \mathcal{E}_6=-\Pi\mathcal{E}_6\Pi^{-1}.$$

From the latter relations we obtain $\Pi=\mathcal{E}_{246}\sim C$. Further, in accordance with $\mathsf{K}=\Pi\mathsf{W}$ for the matrix of the pseudoautomorphism $\mathcal{A}\rightarrow\overline{\mathcal{A}^\star}$ we have $\mathsf{K}=\mathcal{E}_{135}\sim CP$. Finally, for the pseudoantiautomorphisms $\mathcal{A}\rightarrow\overline{\widetilde{\mathcal{A}}}$, $\mathcal{A}\rightarrow\overline{\widetilde{\mathcal{A}^\star}}$ from the definitions $\mathsf{S}=\Pi\mathsf{E}$, $\mathsf{F}=\Pi\mathsf{C}$ it follows that $\mathsf{S}=\mathcal{E}_{25}\sim CT$, $\mathsf{F}=\mathcal{E}_{1346}\sim CPT$. Thus, we come to the following automorphism group:

$$\mathrm{Ext}(\mathbb{C}_6)\simeq\{\mathsf{I},\mathsf{W},\mathsf{E},\mathsf{C},\Pi,\mathsf{K},\mathsf{S},\mathsf{F}\}\simeq\{1,P,T,PT,C,CP,CT,CPT\}\simeq$$
$$\{\mathbf{1}_8,\mathcal{E}_{123456},\mathcal{E}_{456},\mathcal{E}_{123},\mathcal{E}_{246},\mathcal{E}_{135},\mathcal{E}_{25},\mathcal{E}_{1346}\}.$$

The multiplication table of this group is given in Tab. 5. From this table it follows that $\mathrm{Ext}(\mathbb{C}_6)\simeq D_4$, and for the CPT group we have the following isomorphism: $C^{-,+,+,+,+,-,+}\simeq D_4\otimes\mathbb{Z}_2$.

	$\mathbf{1}_8$	$\mathcal{E}_{123456}$	$\mathcal{E}_{456}$	$\mathcal{E}_{123}$	$\mathcal{E}_{246}$	$\mathcal{E}_{135}$	$\mathcal{E}_{25}$	$\mathcal{E}_{1346}$
$\mathbf{1}_8$	$\mathbf{1}_8$	$\mathcal{E}_{123456}$	$\mathcal{E}_{456}$	$\mathcal{E}_{123}$	$\mathcal{E}_{246}$	$\mathcal{E}_{135}$	$\mathcal{E}_{25}$	$\mathcal{E}_{1346}$
$\mathcal{E}_{123456}$	$\mathcal{E}_{123456}$	$-\mathbf{1}_8$	$\mathcal{E}_{123}$	$-\mathcal{E}_{456}$	$-\mathcal{E}_{135}$	$\mathcal{E}_{246}$	$-\mathcal{E}_{1346}$	$\mathcal{E}_{25}$
$\mathcal{E}_{456}$	$\mathcal{E}_{456}$	$-\mathcal{E}_{123}$	$\mathbf{1}_8$	$-\mathcal{E}_{123456}$	$-\mathcal{E}_{25}$	$\mathcal{E}_{1346}$	$-\mathcal{E}_{246}$	$\mathcal{E}_{135}$
$\mathcal{E}_{123}$	$\mathcal{E}_{123}$	$\mathcal{E}_{456}$	$\mathcal{E}_{123456}$	$\mathbf{1}_8$	$\mathcal{E}_{1346}$	$\mathcal{E}_{25}$	$\mathcal{E}_{135}$	$\mathcal{E}_{246}$
$\mathcal{E}_{246}$	$\mathcal{E}_{246}$	$\mathcal{E}_{135}$	$\mathcal{E}_{25}$	$\mathcal{E}_{1346}$	$\mathbf{1}_8$	$\mathcal{E}_{123456}$	$\mathcal{E}_{456}$	$\mathcal{E}_{123}$
$\mathcal{E}_{135}$	$\mathcal{E}_{135}$	$-\mathcal{E}_{246}$	$\mathcal{E}_{1346}$	$-\mathcal{E}_{25}$	$-\mathcal{E}_{123456}$	$\mathbf{1}_8$	$-\mathcal{E}_{123}$	$\mathcal{E}_{456}$
$\mathcal{E}_{25}$	$\mathcal{E}_{25}$	$-\mathcal{E}_{1346}$	$\mathcal{E}_{246}$	$-\mathcal{E}_{135}$	$-\mathcal{E}_{456}$	$\mathcal{E}_{123}$	$-\mathbf{1}_8$	$\mathcal{E}_{123456}$
$\mathcal{E}_{1346}$	$\mathcal{E}_{1346}$	$\mathcal{E}_{25}$	$\mathcal{E}_{135}$	$\mathcal{E}_{246}$	$\mathcal{E}_{123}$	$\mathcal{E}_{456}$	$\mathcal{E}_{123456}$	$\mathbf{1}_8$

Tab. 5: The multiplication table of the $CPT/\mathbb{Z}_2$ group of the field $(1,0)\oplus(0,1)$.

6 The CPT group of the spin-$3/2$ field

In accordance with the general Fermi-scheme of the interlocking representations of $\mathfrak{G}_+$ (see Fig. 2), the field $(3/2,0)\oplus(0,3/2)$ is defined within the following interlocking scheme:

$$\left(\frac{3}{2},0\right)\longleftrightarrow\left(1,\frac{1}{2}\right)\longleftrightarrow\left(\frac{1}{2},1\right)\longleftrightarrow\left(0,\frac{3}{2}\right)\;.$$

A double covering of the representation, associated with the field $(3/2,0)\oplus(0,3/2)$, is realized in the spinspace

$$\mathbb{S}_2\otimes\mathbb{S}_2\otimes\mathbb{S}_2\bigoplus\dot{\mathbb{S}}_2\otimes\dot{\mathbb{S}}_2\otimes\dot{\mathbb{S}}_2, \tag{44}$$

This spinspace is a space of the representation $\mathfrak{C}^{3,0}\oplus\mathfrak{C}^{0,-3}$ of the group $\mathbf{Spin}_+(1,3)$. The algebra

$$\mathbb{C}_2\otimes\mathbb{C}_2\otimes\mathbb{C}_2\bigoplus\overset{*}{\mathbb{C}}_2\otimes\overset{*}{\mathbb{C}}_2\otimes\overset{*}{\mathbb{C}}_2 \tag{45}$$

is associated with the representation $\mathfrak{C}^{3,0}\oplus\mathfrak{C}^{0,-3}$. Spinor representations of the automorphisms, defined on the spinspace (44), are constructed via the Brauer-Weyl representation (40). A spinbasis of the algebra (45) is defined by the following 16×16 matrices:

$$\mathcal{E}_1=\sigma_1\otimes\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2=\begin{bmatrix}0&0&0&0&\mathbf{1}_2&0&0&0\\0&0&0&0&0&\mathbf{1}_2&0&0\\0&0&0&0&0&0&\mathbf{1}_2&0\\0&0&0&0&0&0&0&\mathbf{1}_2\\\mathbf{1}_2&0&0&0&0&0&0&0\\0&\mathbf{1}_2&0&0&0&0&0&0\\0&0&\mathbf{1}_2&0&0&0&0&0\\0&0&0&\mathbf{1}_2&0&0&0&0\end{bmatrix},$$

$$\mathcal{E}_2 = \sigma_3 \otimes \sigma_1 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 = \begin{bmatrix} 0 & 0 & i\mathbf{1}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbf{1}_2 & 0 & 0 & 0 & 0 \\ i\mathbf{1}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\mathbf{1}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i\mathbf{1}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\mathbf{1}_2 \\ 0 & 0 & 0 & 0 & -i\mathbf{1}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i\mathbf{1}_2 & 0 & 0 \end{bmatrix},$$

$$\mathcal{E}_3 = \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbf{1}_2 = \begin{bmatrix} 0 & -\mathbf{1}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{1}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{1}_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{1}_2 & 0 \end{bmatrix},$$

$$\mathcal{E}_4 = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 = \begin{bmatrix} -i\sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i\sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sigma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i\sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i\sigma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i\sigma_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i\sigma_1 \end{bmatrix},$$

$$\mathcal{E}_5 = \sigma_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & -i\mathbf{1}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i\mathbf{1}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i\mathbf{1}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\mathbf{1}_2 \\ i\mathbf{1}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\mathbf{1}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i\mathbf{1}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbf{1}_2 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{E}_6 = \sigma_3 \otimes \sigma_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 = \begin{bmatrix} 0 & 0 & i\mathbf{1}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbf{1}_2 & 0 & 0 & 0 & 0 \\ -i\mathbf{1}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i\mathbf{1}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i\mathbf{1}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\mathbf{1}_2 \\ 0 & 0 & 0 & 0 & i\mathbf{1}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\mathbf{1}_2 & 0 & 0 \end{bmatrix},$$

$$
\mathcal{E}_7=\sigma_3\otimes\sigma_3\otimes\sigma_2\otimes\mathbf{1}_2=\begin{bmatrix}
0 & i\mathbf{1}_2 & 0 & 0 & 0 & 0 & 0 & 0\\
-i\mathbf{1}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -i\mathbf{1}_2 & 0 & 0 & 0 & 0\\
0 & 0 & i\mathbf{1}_2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -i\mathbf{1}_2 & 0 & 0\\
0 & 0 & 0 & 0 & i\mathbf{1}_2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i\mathbf{1}_2\\
0 & 0 & 0 & 0 & 0 & 0 & -i\mathbf{1}_2 & 0
\end{bmatrix},
$$

$$
\mathcal{E}_8=\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_2=\begin{bmatrix}
-i\sigma_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & i\sigma_2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & i\sigma_2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -i\sigma_2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & i\sigma_2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -i\sigma_2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -i\sigma_2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i\sigma_2
\end{bmatrix}.
$$

Using this spinbasis, we construct CPT group for the field $(3/2,0)\oplus(0,3/2)$. At first, the matrix of the automorphism $\mathcal{A}\rightarrow\mathcal{A}^\star$ has the form

$$
\mathsf{W}=\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3\mathcal{E}_4\mathcal{E}_5\mathcal{E}_6\mathcal{E}_7\mathcal{E}_8=\mathcal{E}_{12345678}\sim P.
$$

Further, since

$$
\mathcal{E}_1^{\mathsf{T}}=\mathcal{E}_1,\quad \mathcal{E}_2^{\mathsf{T}}=\mathcal{E}_2,\quad \mathcal{E}_3^{\mathsf{T}}=\mathcal{E}_3,\quad \mathcal{E}_4^{\mathsf{T}}=\mathcal{E}_4,\quad \mathcal{E}_5^{\mathsf{T}}=-\mathcal{E}_5,\quad \mathcal{E}_6^{\mathsf{T}}=-\mathcal{E}_6,\quad \mathcal{E}_7^{\mathsf{T}}=-\mathcal{E}_7,\quad \mathcal{E}_8^{\mathsf{T}}=-\mathcal{E}_8,
$$

then in accordance with $\widetilde{\mathsf{A}}=\mathsf{E}\mathsf{A}^{\mathsf{T}}\mathsf{E}^{-1}$ we have

$$
\mathcal{E}_1=\mathsf{E}\mathcal{E}_1\mathsf{E}^{-1},\quad \mathcal{E}_2=\mathsf{E}\mathcal{E}_2\mathsf{E}^{-1},\quad \mathcal{E}_3=\mathsf{E}\mathcal{E}_3\mathsf{E}^{-1},\quad \mathcal{E}_4=\mathsf{E}\mathcal{E}_4\mathsf{E}^{-1},
$$

$$
\mathcal{E}_5=-\mathsf{E}\mathcal{E}_5\mathsf{E}^{-1},\quad \mathcal{E}_6=-\mathsf{E}\mathcal{E}_6\mathsf{E}^{-1},\quad \mathcal{E}_7=-\mathsf{E}\mathcal{E}_7\mathsf{E}^{-1},\quad \mathcal{E}_8=-\mathsf{E}\mathcal{E}_8\mathsf{E}^{-1}.
$$

Hence it follows that E commutes with $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$ and anticommutes with $\mathcal{E}_5$, $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$, that is, $\mathsf{E}=\mathcal{E}_{5678}\sim T$. From the definition $\mathsf{C}=\mathsf{E}\mathsf{W}$ we find that a matrix of the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}^\star}$ has the form $\mathsf{C}=\mathcal{E}_{1234}\sim PT$. The basis $\{\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3,\mathcal{E}_4,\mathcal{E}_5,\mathcal{E}_6,\mathcal{E}_7,\mathcal{E}_8\}$ contains both complex and real matrices:

$$
\mathcal{E}_1^*=\mathcal{E}_1,\quad \mathcal{E}_2^*=-\mathcal{E}_2,\quad \mathcal{E}_3^*=\mathcal{E}_3,\quad \mathcal{E}_4^*=-\mathcal{E}_4,\quad \mathcal{E}_5^*=-\mathcal{E}_5,\quad \mathcal{E}_6^*=-\mathcal{E}_6,\quad \mathcal{E}_7^*=-\mathcal{E}_7,\quad \mathcal{E}_8^*=\mathcal{E}_8.
$$

Therefore, from $\overline{\mathsf{A}}=\Pi\mathsf{A}^*\Pi^{-1}$ we have

$$
\mathcal{E}_1=\Pi\mathcal{E}_1\Pi^{-1},\quad \mathcal{E}_2=-\Pi\mathcal{E}_2\Pi^{-1},\quad \mathcal{E}_3=\Pi\mathcal{E}_3\Pi^{-1},\quad \mathcal{E}_4=-\Pi\mathcal{E}_4\Pi^{-1},
$$

$$
\mathcal{E}_5=-\Pi\mathcal{E}_5\Pi^{-1},\quad \mathcal{E}_6=-\Pi\mathcal{E}_6\Pi^{-1},\quad \mathcal{E}_7=-\Pi\mathcal{E}_7\Pi^{-1},\quad \mathcal{E}_8=\Pi\mathcal{E}_8\Pi^{-1}.
$$

From the latter relations we obtain $\Pi=\mathcal{E}_{24567}\sim C$. Further, in accordance with $\mathsf{K}=\Pi\mathsf{W}$ for the matrix of the pseudoautomorphism $\mathcal{A}\rightarrow\overline{\mathcal{A}^\star}$ we have $\mathsf{K}=\mathcal{E}_{138}\sim CP$. Finally, for the pseudoantiautomorphisms $\mathcal{A}\rightarrow\overline{\widetilde{\mathcal{A}}}$ and $\mathcal{A}\rightarrow\overline{\widetilde{\mathcal{A}^\star}}$ from the definitions $\mathsf{S}=\Pi\mathsf{E}$ and $\mathsf{F}=\Pi\mathsf{C}$ it follows that $\mathsf{S}=\mathcal{E}_{248}\sim CT$, $\mathsf{F}=\mathcal{E}_{13567}\sim CPT$. Thus, we come to the following automorphism group:

$$
\mathrm{Ext}(\mathbb{C}_8)\simeq\{\mathsf{I},\mathsf{W},\mathsf{E},\mathsf{C},\Pi,\mathsf{K},\mathsf{S},\mathsf{F}\}\simeq\{1,P,T,PT,C,CP,CT,CPT\}\simeq
$$
$$
\{\mathbf{1}_{16},\mathcal{E}_{12345678},\mathcal{E}_{5678},\mathcal{E}_{1234},\mathcal{E}_{24567},\mathcal{E}_{138},\mathcal{E}_{246},\mathcal{E}_{13567}\}.
$$

The multiplication table of this group is given in Tab. 6. From this table it follows that $\mathrm{Ext}(\mathbb{C}_8)\simeq D_4$, and for the CPT group we have the following isomorphism: $C^{-,-,+,+,+,+,+}\simeq D_4\otimes\mathbb{Z}_2$.

	$\mathbf{1}_{16}$	W	$\mathcal{E}_{5678}$	$\mathcal{E}_{1234}$	$\mathcal{E}_{24567}$	$\mathcal{E}_{138}$	$\mathcal{E}_{248}$	$\mathcal{E}_{13567}$
$\mathbf{1}_{16}$	$\mathbf{1}_{16}$	W	$\mathcal{E}_{5678}$	$\mathcal{E}_{1234}$	$\mathcal{E}_{24567}$	$\mathcal{E}_{138}$	$\mathcal{E}_{248}$	$\mathcal{E}_{13567}$
W	W	$-\mathbf{1}_{16}$	$-\mathcal{E}_{1234}$	$\mathcal{E}_{5678}$	$\mathcal{E}_{138}$	$-\mathcal{E}_{24567}$	$-\mathcal{E}_{13567}$	$-\mathcal{E}_{248}$
$\mathcal{E}_{5678}$	$\mathcal{E}_{5678}$	$-\mathcal{E}_{1234}$	$-\mathbf{1}_{16}$	W	$\mathcal{E}_{248}$	$-\mathcal{E}_{13567}$	$-\mathcal{E}_{24567}$	$\mathcal{E}_{138}$
$\mathcal{E}_{1234}$	$\mathcal{E}_{1234}$	$\mathcal{E}_{5678}$	W	$\mathbf{1}_{16}$	$\mathcal{E}_{13567}$	$\mathcal{E}_{248}$	$\mathcal{E}_{138}$	$\mathcal{E}_{24567}$
$\mathcal{E}_{24567}$	$\mathcal{E}_{24567}$	$-\mathcal{E}_{138}$	$-\mathcal{E}_{248}$	$\mathcal{E}_{13567}$	$\mathbf{1}_{16}$	$-$W	$-\mathcal{E}_{5678}$	$\mathcal{E}_{1234}$
$\mathcal{E}_{138}$	$\mathcal{E}_{138}$	$\mathcal{E}_{24567}$	$\mathcal{E}_{13567}$	$\mathcal{E}_{248}$	W	$\mathbf{1}_{16}$	$\mathcal{E}_{1234}$	$\mathcal{E}_{5678}$
$\mathcal{E}_{248}$	$\mathcal{E}_{248}$	$\mathcal{E}_{13567}$	$\mathcal{E}_{24567}$	$\mathcal{E}_{138}$	$\mathcal{E}_{5678}$	$\mathcal{E}_{1234}$	$\mathbf{1}_{16}$	W
$\mathcal{E}_{13567}$	$\mathcal{E}_{13567}$	$-\mathcal{E}_{248}$	$-\mathcal{E}_{138}$	$\mathcal{E}_{24567}$	$\mathcal{E}_{1234}$	$-\mathcal{E}_{5678}$	$-$W	$\mathbf{1}_{16}$

Tab. 6: The multiplication table of the $CPT/\mathbb{Z}_2$ group of the field $(3/2,0)\oplus(0,3/2)$.

7 CPT groups of the tensor fields

As it is shown in the section 3 double coverings of the representations associated with the tensor fields are constructed within the product $\mathbb{C}_2\otimes\mathbb{C}_2\otimes\cdots\otimes\mathbb{C}_2\bigotimes\overset{*}{\mathbb{C}}_2\otimes\overset{*}{\mathbb{C}}_2\otimes\cdots\otimes\overset{*}{\mathbb{C}}_2$, where we have k algebras $\mathbb{C}_2$ and r algebras $\overset{*}{\mathbb{C}}_2$. A relation between the number l (a weight of the representation in the Van der Waerden basis (37)) and the numbers k and r is given by the formula

$$l=\frac{k-r}{2}. \tag{46}$$

It is easy to see that a central row in the scheme shown on the Fig. 1,

$$(0,0)\ -\ \left(\frac{1}{2},\frac{1}{2}\right)\ -\ (1,1)\ —\ \cdots\ —\ \left(\frac{s}{2},\frac{s}{2}\right)\ —\ \cdots \tag{47}$$

in virtue of (46) is equivalent to the following row:

$$[0,0]\ -\ [0,0]\ -\ [0,0]\ —\ \cdots\ —\ [0,0]\ —\ \cdots$$

Analogously, the row shown on the Fig. 2,

$$\left(\frac{1}{2},0\right)\ —\ \left(1,\frac{1}{2}\right)\ —\ \cdots\ -\ \left(\frac{2s+1}{4},\frac{2s-1}{4}\right)\ -\ \cdots \tag{48}$$

is equivalent to

$$\left[\frac{1}{2},0\right]\ -\ \left[\frac{1}{2},0\right]\ —\ \cdots\ —\ \left[\frac{1}{2},0\right]\ —\ \cdots$$

Therefore, all the representations of $\mathbf{Spin}_+(1,3)$ can be divided on the equivalent rows which we show on the Fig. 3 and Fig. 4. On the other hand, the row (47) corresponds to the following chain

$$
\begin{array}{ccccccccccc}
 & & & & & & & & [s,0] & \text{—} & \cdots \\
 & & & & & & & & | & & \\
 & & & & & & & & \vdots & & \\
 & & & & & & & & | & & \\
 & & & & [2,0] & \text{—} & \cdots & \text{—} & [2,0] & \text{—} & \cdots \\
 & & & & | & & & & | & & \\
 & & [1,0] & - & [1,0] & \text{—} & \cdots & \text{—} & [1,0] & \text{—} & \cdots \\
 & & | & & | & & & & | & & \\
[0,0] & - & [0,0] & - & [0,0] & \text{—} & \cdots & \text{—} & [0,0] & \text{—} & \cdots \\
 & & | & & | & & & & | & & \\
 & & [0,1] & - & [0,1] & \text{—} & \cdots & \text{—} & [0,1] & \text{—} & \cdots \\
 & & & & | & & & & | & & \\
 & & & & [0,2] & \text{—} & \cdots & \text{—} & [0,2] & \text{—} & \cdots \\
 & & & & & & & & | & & \\
 & & & & & & & & \vdots & & \\
 & & & & & & & & | & & \\
 & & & & & & & & [0,s] & \text{—} & \cdots
\end{array}
$$

Fig. 3: Integer spin representations of the group $\mathbf{Spin}_+(1,3)$.

of the algebras:

$$
\mathbf{1} \longrightarrow \mathbb{C}_2 \otimes \overset{*}{\mathbb{C}}_2 \longrightarrow \mathbb{C}_2 \otimes \mathbb{C}_2 \bigotimes \overset{*}{\mathbb{C}}_2 \otimes \overset{*}{\mathbb{C}}_2 \longrightarrow \ldots \longrightarrow
$$
$$
\longrightarrow \underbrace{\mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2}_{s \text{ times}} \bigotimes \underbrace{\overset{*}{\mathbb{C}}_2 \otimes \overset{*}{\mathbb{C}}_2 \otimes \cdots \otimes \overset{*}{\mathbb{C}}_2}_{s \text{ times}} \longrightarrow \ldots
$$

In its turn, the row (48) corresponds to the chain

$$
\mathbb{C}_2 \longrightarrow \mathbb{C}_2 \otimes \mathbb{C}_2 \bigotimes \overset{*}{\mathbb{C}}_2 \longrightarrow \ldots \longrightarrow \underbrace{\mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2}_{(2s+1)/2 \text{ times}} \bigotimes \underbrace{\overset{*}{\mathbb{C}}_2 \otimes \overset{*}{\mathbb{C}}_2 \otimes \cdots \otimes \overset{*}{\mathbb{C}}_2}_{(2s-1)/2 \text{ times}} \longrightarrow \ldots
$$

Moreover, these chains induces the following chains of the spinspaces:

$$
\mathbb{S}_0 \longrightarrow \mathbb{S}_4 \longrightarrow \mathbb{S}_{16} \longrightarrow \ldots \longrightarrow \mathbb{S}_{2^{2s}} \longrightarrow \ldots
$$

and

$$
\mathbb{S}_2 \longrightarrow \mathbb{S}_8 \longrightarrow \ldots \longrightarrow \mathbb{S}_{2^{2s}} \longrightarrow \ldots
$$

Thus, the row (47) (or (48)) induces a sequence of the fields of the spin 0 (or 1/2) realized in the spinspaces of different dimensions. In general case presented on the Fig. 3 and Fig. 4 we have sequences of the fields of the same spin realized in the different representation spaces of $\mathbf{Spin}_+(1,3)$. One can say that this situation corresponds to particles of the same spin with different masses, like proton $\longrightarrow$ electron $\longrightarrow$ ... (spin 1/2). With the aim to give more detailed explanation for this statement let us consider a Gel'fand-Yaglom mass spectrum formula [21]:

$$
\mu^{(l)} = \frac{\kappa}{l + \frac{1}{2}} = \frac{2\kappa}{2l+1}, \tag{49}
$$

$$
\begin{tikzcd}
 & & & {[s,0]} \arrow[r, dash] \arrow[d, dash] & \cdots \\
 & & & \vdots \arrow[d, dash] & \\
 & {\left[\frac{3}{2},0\right]} \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & {\left[\frac{3}{2},0\right]} \arrow[r, dash] \arrow[d, dash] & \cdots \\
{\left[\frac{1}{2},0\right]} \arrow[r, dash] \arrow[d, dash] & {\left[\frac{1}{2},0\right]} \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & {\left[\frac{1}{2},0\right]} \arrow[r, dash] \arrow[d, dash] & \cdots \\
{\left[0,\frac{1}{2}\right]} \arrow[r, dash] & {\left[0,\frac{1}{2}\right]} \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & {\left[0,\frac{1}{2}\right]} \arrow[r, dash] \arrow[d, dash] & \cdots \\
 & {\left[0,\frac{3}{2}\right]} \arrow[r, dash] & \cdots \arrow[r, dash] & {\left[0,\frac{3}{2}\right]} \arrow[r, dash] \arrow[d, dash] & \cdots \\
 & & & \vdots \arrow[d, dash] & \\
 & & & {[0,s]} \arrow[r, dash] & \cdots
\end{tikzcd}
$$

Fig. 4: Half-integer spin representations of the group $\mathbf{Spin}_+(1,3)$.

where the mass $\mu^{(l)}$ corresponds the spin l, κ is a constant. It is easy to see that the denominator $2l+1$ in (49) is equal to a dimensionality of the representation space $\mathrm{Sym}_{(k,0)}$ corresponding to the field $\boldsymbol{\psi}(\boldsymbol{\alpha})$ of type $(l,0)$ (or $(0,\dot{l})$ and $\mathrm{Sym}_{(0,r)}$). For the tensor fields $\boldsymbol{\psi}(\boldsymbol{\alpha})$ of type $(l\dot{l})$ we have

$$\mu^{(s)} = \frac{\kappa}{(k+1)(r+1)}, \tag{50}$$

where $s = |k-r|/2$ is a spin of the field $\boldsymbol{\psi}(\boldsymbol{\alpha})$. In this case, the denominator in (50) is equal to a dimensionality of the representation space $\mathrm{Sym}_{(k,r)}$ corresponding to the tensor field. Mass spectrum formulas (49) and (50) give a relationship between dimensions of the representation spaces of $\mathbf{Spin}_+(1,3)$ and particle masses. From the formula (50) it follows directly that on the parallel rows presented on the Fig. 3 and Fig. 4 we have particles of the same spin with different masses. When $l \to \infty$ (or $(k+1)(r+1) \to \infty$) we come to particles with zero mass (like a photon). In this case, finite-dimensional representation spaces $\mathrm{Sym}_{(k,0)}$ and $\mathrm{Sym}_{(k,r)}$ should be replaced by a Hilbert space, and such (massless) particles should be described within principal series of infinite-dimensional representations of the group $\mathbf{Spin}_+(1,3)$.

CPT groups of the tensor fields are constructed via the same procedure that considered in the sections 4–6. For example, the tensor field of the spin 1/2 corresponding to the interlocking scheme

$$\left(\frac{3}{2},1\right) \longleftrightarrow \left(1,\frac{3}{2}\right)$$

(which is equivalent to $(1/2,0)\oplus(0,1/2)$), is constructed within the algebra

$$\mathbb{C}_2\otimes\mathbb{C}_2\otimes\mathbb{C}_2\bigotimes\overset{*}{\mathbb{C}}_2\bigoplus\overset{*}{\mathbb{C}}_2\otimes\overset{*}{\mathbb{C}}_2\bigotimes\mathbb{C}_2\otimes\mathbb{C}_2\otimes\mathbb{C}_2. \tag{51}$$

This algebra induces the spinspace

$$\mathbb{S}_2\otimes\mathbb{S}_2\otimes\mathbb{S}_2\bigotimes\dot{\mathbb{S}}_2\otimes\dot{\mathbb{S}}_2\bigoplus\dot{\mathbb{S}}_2\otimes\dot{\mathbb{S}}_2\bigotimes\mathbb{S}_2\otimes\mathbb{S}_2\otimes\mathbb{S}_2\simeq\mathbb{S}_{64}.$$

The spinbasis of the algebra (51) is defined by the following 64×64 matrices:

$$\begin{array}{ll}
\mathcal{E}_1=\sigma_1\otimes\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2, & \mathcal{E}_2=\sigma_3\otimes\sigma_1\otimes\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2,\\
\mathcal{E}_3=\sigma_3\otimes\sigma_3\otimes\sigma_1\otimes\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2, & \mathcal{E}_4=\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_1\otimes\mathbf{1}_2\otimes\mathbf{1}_2,\\
\mathcal{E}_5=\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_1\otimes\mathbf{1}_2, & \mathcal{E}_6=\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_1,\\
\mathcal{E}_7=\sigma_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2, & \mathcal{E}_8=\sigma_3\otimes\sigma_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2,\\
\mathcal{E}_9=\sigma_3\otimes\sigma_3\otimes\sigma_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2, & \mathcal{E}_{10}=\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2,\\
\mathcal{E}_{11}=\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_2\otimes\mathbf{1}_2, & \mathcal{E}_{12}=\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_2.
\end{array}$$

The extended automorphism group $\mathrm{Ext}(\mathbb{C}_{12})$ can be derived from this spinbasis via the same calculations that presented in the sections 4–6.

8 Summary

We have presented a group theoretical method for description of discrete symmetries of the fields $\boldsymbol{\psi}(\boldsymbol{\alpha})=\langle x,\mathfrak{g}\,|\boldsymbol{\psi}\rangle$, where $x\in T_4$ and $\mathfrak{g}\in\mathbf{Spin}_+(1,3)$, in terms of involutive automorphisms of the subgroup $\mathbf{Spin}_+(1,3)\simeq\mathrm{SU}(2)\otimes\mathrm{SU}(2)$. We have shown that an extended automorphism group $\mathrm{Ext}(\mathbb{C}_n)$, where $\mathbb{C}_n$ is a complex Clifford algebra, lead to CPT groups of the fields $\boldsymbol{\psi}(\boldsymbol{\alpha})=\langle x,\mathfrak{g}\,|\boldsymbol{\psi}\rangle$ of any spin defined on the representation spaces (spinspaces) of $\mathbf{Spin}_+(1,3)$. We considered in detail CPT groups for the fields of the type $(l,0)\oplus(0,l)$ (for example, $(1/2,0)\oplus(0,1/2)$, $(1,0)\oplus(0,1)$ and $(3/2,0)\oplus(0,3/2)$). Also we discussed CPT groups for the fields of tensor type and their relations to particles of the same spin with different masses. It would be interesting to consider extended automorphism groups $\mathrm{Ext}(C\!\ell_{p,q})$, where $C\!\ell_{p,q}$ is a real Clifford algebra, defined on the real representations of $\mathbf{Spin}_+(1,3)$. It would be interesting also to consider CPT groups for the fields $\boldsymbol{\psi}(\boldsymbol{\alpha})=\langle x,\mathfrak{q}\,|\boldsymbol{\psi}\rangle$ on the de Sitter group, where $x\in T_5$ and $\mathfrak{q}\in\mathbf{Spin}_+(1,4)\simeq\mathrm{Sp}(1,1)$, and for the fields $\boldsymbol{\psi}(\boldsymbol{\alpha})=\langle x,\mathfrak{c}\,|\boldsymbol{\psi}\rangle$ on the conformal group, where $x\in T_6$ and $\mathfrak{c}\in\mathbf{Spin}_+(2,6)\simeq\mathrm{SU}(2,2)$. Our next paper will be devoted to these questions.

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